Homework 9

This homework is due on Friday, April 1, 2022, at 11:59PM. Self-grades and HW Resubmissions are due on the following Friday, April 8, 2022, at 11:59PM.

1. **Reading Lecture Notes**

   Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: Note 13 Note 15.

   (a) Consider $A \in \mathbb{R}^{n\times n}$ where the columns of A (denoted by $\vec{a}_k, 1 \le k \le n$) are orthonormal. **What does the least squares solution $(A^\top A)^{-1}A^\top \vec{y}$ simplify to?**

   Solution: Since the columns of A are orthonormal, then $(A^\top A)^{-1} = I$. If we write $A = \begin{bmatrix}\vec{a}_1 & \dots & \vec{a}_n\end{bmatrix}$ then $A^\top = \begin{bmatrix}\vec{a}_1^\top \\ \vdots \\ \vec{a}_n^\top\end{bmatrix}$ and finally $A^\top\vec{y} = \begin{bmatrix}\vec{a}_1^\top\vec{y} \\ \vdots \\ \vec{a}_n^\top\vec{y}\end{bmatrix}$. So, our final expression is

   $$(A^\top A)^{-1}A^\top\vec{y} = \begin{bmatrix}\vec{a}_1^\top\vec{y} \\ \vdots \\ \vec{a}_n^\top\vec{y}\end{bmatrix}. \tag{1}$$

   (b) Suppose we have two vectors $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix} \in \mathbb{R}^2$. Then, we use Gram-Schmidt orthonormalization to construct $\vec{q}_1$ and $\vec{q}_2$. **Are $\vec{q}_1$ and $\vec{q}_2$ the only vectors that form an orthogonal basis for** $\text{Span}(\vec{v}_1, \vec{v}_2)$**?**

   Solution: No, in fact there are infinitely many vectors that make an orthogonal basis for $\text{Span}(\vec{v}_1, \vec{v}_2)$! We can take $\vec{q}_1$ and $\vec{q}_2$ and rotate them by the same angle. That is, define $\vec{u}_1(\theta) = R_\theta\vec{q}_1$ and $\vec{u}_2(\theta) = R_\theta\vec{q}_2$ where R_θ is a matrix that rotates a 2-D vector by an angle θ. So, $\vec{u}_1(\theta)$ and $\vec{u}_2(\theta)$ form an orthogonal basis for $\text{Span}(\vec{v}_1, \vec{v}_2)$ for all values of θ.

   (c) Give a brief outline for how you would compute the Schur decomposition (i.e. upper-triangularization) of some general square matrix under the assumption that all eigenvalues are real.

   Solution:

   - Compute an eigenvalue and eigenvector of your matrix, A.
   - Create an orthonormal matrix with your eigenvector as the first vector: V.
   - Transform your matrix: $V^\top AV$, then take the $n-1\times n-1$ sub-matrix ignoring the first row and column with the eigenvalue inside. Iterate from the beginning.
   - Stitch all matrices V generated along the way into one single matrix so that we can express $A = VTV^\top$ where T is upper triangular.

   (d) What happens when we upper-triangularize a symmetric matrix?

   Solution: For a symmetric matrix, it turns out that the Schur decomposition is the same as a diagonalization or eigendecomposition. We just get a diagonal matrix as our upper-triangular matrix.

2. Gram-Schmidt Basics

(a) **Use Gram-Schmidt to find a matrix U whose columns form an orthonormal basis for the column space of V.**

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Solution: We start with the columns of V as our basis for the column space of V, and we want to find an orthonormal basis for this same space using Gram-Schmidt. For notational convenience, define

$$V = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}, U = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \tag{3}$$

We summarize the first few steps of the Gram-Schmidt algorithm as follows:

i. $\vec{u}_1' = \vec{v}_1; \qquad \vec{u}_1 = \frac{\vec{u}_1'}{\|\vec{u}_1'\|}$.

ii. $\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1; \qquad \vec{u}_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|}$.

iii. $\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2; \qquad \vec{u}_3 = \frac{\vec{u}_3'}{\|\vec{u}_3'\|}$.

For the column space of V, this is

i. $\vec{u}_1' = \vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$. Since $\vec{u}_1'$ is already normalized, we simply set $\vec{u}_1 = \vec{u}_1'$.

ii.

$$\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \implies \vec{u}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

iii.

$$\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \implies \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{5}$$

Thus, the matrix U is given by

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{6}$$

(b) Show that you get the same resulting vector when you project $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ onto the columns of V as you do when you project onto the columns of U, i.e. **show that**

$$V(V^\top V)^{-1} V^\top \vec{w} = U(U^\top U)^{-1} U^\top \vec{w}. \tag{7}$$

Feel free to use `NumPy` for the projection onto the columns of V, but compute the projection onto the columns of U by hand. Comment on whether projecting upon the V or U basis is computationally more efficient. *(HINT: Which of these matrices allow us to circumvent the matrix inversion in the projection formula?)*

Solution: Note that whatever basis we use for a subspace, when we project a vector onto that subspace, we get the same vector. For example, when we project the vector $\vec{w} = \begin{bmatrix}1 & -1 & 0 & -1 & 0\end{bmatrix}^\top$ onto the subspace using the V basis, we get

$$V(V^\top V)^{-1}V^\top \vec{w} = \begin{bmatrix}1&1&1\\0&1&1\\0&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&3&3\\1&3&5\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0&0\\1&1&1&0&0\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}1\\-1\\0\\-1\\0\end{bmatrix} \tag{8}$$

$$= \begin{bmatrix}1&1&1\\0&1&1\\0&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{3}{2}&-\frac{1}{2}&0\\-\frac{1}{2}&1&-\frac{1}{2}\\0&-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} \tag{9}$$

$$= \begin{bmatrix}1&1&1\\0&1&1\\0&1&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{3}{2}\\0\\-\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \tag{10}$$

$$U(U^\top U)^{-1}U^\top \vec{w} = \begin{bmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0\\0&0&\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0\\0&0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\-1\\0\\-1\\0\end{bmatrix} \tag{11}$$

$$= \begin{bmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}&0\\0&0&\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \tag{12}$$

Note however that the projection using the U basis was much simpler. Since $U^\top U$ is the identity, we didn't need to do a matrix inversion.

3. Upper Triangularization

In this problem, you need to upper-triangularize the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \tag{13}$$

The eigenvalues of this matrix A are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = -4$. We want to express A as

$$A = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot \begin{bmatrix} \vec{x}^\top \\ \vec{y}^\top \\ \vec{z}^\top \end{bmatrix} \tag{14}$$

where $\vec{x}, \vec{y}, \vec{z}$ are orthonormal. Your goal in this problem is to compute $\vec{x}, \vec{y}, \vec{z}$ so that they satisfy the above relationship for some constants a, b, c.

Here are some potentially useful facts that we have gathered to save you some computations, you'll have to grind out the rest yourself.

$$\begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix}. \tag{15}$$

We also know that

$$\left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{16}$$

is an orthonormal basis, and

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -2 \end{bmatrix}. \tag{17}$$

We also know that $\begin{bmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & -2 \end{bmatrix}$ has eigenvalues 2 and -4. The normalized eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$. A vector which is orthogonal to that normalized eigenvector and is itself normalized is $\begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$.

Based on the above information, compute $\vec{x}, \vec{y}, \vec{z}$. Show your work.

You don't have to compute the constants a, b, c in the interests of time.

Solution:

Following the Schur Decomposition procedure, we first find an eigenvalue and a corresponding eigenvector of A. From the given information from the problem:

$$A \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \tag{18}$$

We can conclude that

$$\vec{x} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \tag{19}$$

is an eigenvector of A with eigenvalue $\lambda_1 = 2$. Using the given orthonormal basis including eigenvector $\vec{x}$, we define

$$Q := \begin{bmatrix}\vec{q} & \widetilde{Q}\end{bmatrix} \qquad \text{where} \qquad \vec{q} = \vec{x} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\\ 0\end{bmatrix}, \widetilde{Q} = \begin{bmatrix}\frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & 0\\ 0 & 1\end{bmatrix} \tag{20}$$

From the Schur Decomposition procedure, we have

$$Q^\top A Q = \begin{bmatrix}\lambda_1 & \vec{q}^\top A\widetilde{Q}\\ \vec{0}_{n-1} & \widetilde{Q}^\top A\widetilde{Q}\end{bmatrix} \tag{21}$$

$$= \begin{bmatrix}\lambda_1 & \vec{\widetilde{a}}_{12}^\top\\ \vec{0}_{n-1} & \widetilde{A}_{22}\end{bmatrix}. \tag{22}$$

where

$$\vec{\widetilde{a}}_{12}^\top := \vec{q}^\top A\widetilde{Q} \qquad \text{and} \qquad \widetilde{A}_{22} := \widetilde{Q}^\top A\widetilde{Q}. \tag{23}$$

Using the given computation, We can compute

$$\widetilde{A}_{22} = \widetilde{Q}^\top A\widetilde{Q} = \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1 & 2\\ 3 & -1 & 6\\ -2 & 2 & -2\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & -2\sqrt{2}\\ -2\sqrt{2} & -2\end{bmatrix}. \tag{24}$$

Since $\widetilde{A}_{22}$ is 2×2 matrix, recursively applying Schur Decomposition to $\widetilde{A}_{22}$ would give

$$\widetilde{A}_{22} = P\widetilde{T}P^\top \tag{25}$$

Where P is orthonormal and $\widetilde{T}$ is upper-triangular. From the given information, $\widetilde{A}_{22}$ has following eigenvalues and orthonormal eigenvectors:

$$\lambda_1 = 2, \quad \vec{u}_{\lambda_1} = \begin{bmatrix}-\frac{\sqrt{6}}{3}\\ \frac{\sqrt{3}}{3}\end{bmatrix}, \quad \left\|\vec{u}_{\lambda_1}\right\|_2 = 1 \tag{26}$$

$$\lambda_2 = -4, \quad \vec{u}_{\lambda_2} = \begin{bmatrix}\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3}\end{bmatrix}, \quad \left\|\vec{u}_{\lambda_2}\right\|_2 = 1 \tag{27}$$

$$\vec{u}_{\lambda_1} \perp \vec{u}_{\lambda_2} \tag{28}$$

Therefore, we can set $P, \widetilde{T}$ as following:

$$P = \begin{bmatrix}\vec{u}_{\lambda_1} & \vec{u}_{\lambda_2}\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3}\end{bmatrix} \tag{29}$$

$$\widetilde{T} = \begin{bmatrix}2 & 0\\ 0 & -4\end{bmatrix} \tag{30}$$

Knowing P and $\widetilde{T}$, we can find the resulting upper triangularization of A:

$$A = \underbrace{\begin{bmatrix}\vec{q} & \widetilde{Q}P\end{bmatrix}}_{:=U}\underbrace{\begin{bmatrix}\lambda_1 & \vec{\widetilde{a}}_{12}^\top P\\ \vec{0}_{n-1} & \widetilde{T}\end{bmatrix}}_{:=T}\underbrace{\begin{bmatrix}\vec{q}^\top\\ (\widetilde{Q}P)^\top\end{bmatrix}}_{=U^\top}. \tag{31}$$

In order to find $\vec{x}, \vec{y}, \vec{z}$,

$$U = \begin{bmatrix}\vec{x} & \vec{y} & \vec{z}\end{bmatrix} = \begin{bmatrix}\vec{q} & \widetilde{Q}P\end{bmatrix} = \begin{bmatrix}\vec{q} & \widetilde{Q}\vec{u}_{\lambda_1} & \widetilde{Q}\vec{u}_{\lambda_2}\end{bmatrix} \tag{32}$$

Using the above information, we can finally compute $\vec{y}$ and $\vec{z}$.

$$\vec{y} = \widetilde{Q}\vec{u}_{\lambda_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \tag{33}$$

$$\vec{z} = \widetilde{Q}\vec{u}_{\lambda_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix} \tag{34}$$

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix} \tag{35}$$

Finally, we can find the values a, b, c by

$$U^\top AU = T = \begin{bmatrix} \lambda_1 & a & b \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{36}$$

$$U^\top AU = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4\sqrt{3} \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \tag{37}$$

Hence, we have

$$a = 0, \quad b = 4\sqrt{3}, \quad c = 0 \tag{38}$$

which completes our upper triangularization.

4. Using Upper-Triangularization to Solve Differential Equations

You know that for any square matrix A with real eigenvalues, there exists a real matrix V with orthonormal columns and a real upper triangular matrix R so that $A = VRV^\top$. In particular, to set notation explicitly:

$$V = \begin{bmatrix}\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\end{bmatrix} \tag{39}$$

$$R = \begin{bmatrix}\vec{r}_1^\top \\ \vec{r}_2^\top \\ \vdots \\ \vec{r}_n^\top\end{bmatrix} \tag{40}$$

where the rows of the upper-triangular R look like

$$\vec{r}_1^\top = \begin{bmatrix}\lambda_1 & r_{1,2} & r_{1,3} & \ldots & r_{1,n}\end{bmatrix} \tag{41}$$

$$\vec{r}_2^\top = \begin{bmatrix}0, \lambda_2, r_{2,3}, r_{2,4}, \ldots & r_{2,n}\end{bmatrix} \tag{42}$$

$$\vec{r}_i^\top = \begin{bmatrix}\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, \lambda_i, r_{i,i+1}, r_{i,i+2}, \ldots, r_{i,n}\end{bmatrix} \tag{43}$$

$$\vec{r}_n^\top = \begin{bmatrix}\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \lambda_n\end{bmatrix} \tag{44}$$

where the λ_i are the eigenvalues of A.

Suppose our goal is to solve the n-dimensional system of differential equations written out in vector/matrix form as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t), \tag{45}$$

$$\vec{x}(0) = \vec{x}_0, \tag{46}$$

where $\vec{x}_0$ is a specified initial condition and $\vec{u}(t)$ is a given vector of functions of time.

Assume that the V and R have already been computed and are accessible to you using the notation above.

Assume that you have access to a function `ScalarSolve`$(\lambda, y_0, \breve{u})$ that takes a real number λ, a real number y_0, and a real-valued function of time $\breve{u}$ as inputs and returns a real-valued function of time that is the solution to the scalar differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = \lambda y(t) + \breve{u}(t) \tag{47}$$

with initial condition $y(0) = y_0$.

Also assume that you can do regular arithmetic using real-valued functions and it will do the right thing. So if u is a real-valued function of time, and g is also a real-valued function of time, then $5u + 6g$ will be a real valued function of time that evaluates to $5u(t) + 6g(t)$ at time t.

Use V, R to construct a procedure for solving this differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t), \tag{48}$$

$$\vec{x}(0) = \vec{x}_0, \tag{49}$$

for $\vec{x}(t)$ by filling in the following template in the spots marked $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$.

NOTE: It will be useful to upper triangularize A by change of basis to get a differential equation in terms of R instead of A.

(HINT: The process here should be similar to diagonalization with some modifications. Start from the last row of the system and work your way up to understand the algorithm.)

1: $\vec{\widetilde{x}}_0 = V^\top \vec{x}_0$ ▷ Change the initial condition to be in V-coordinates
2: $\vec{\widetilde{u}} = V^\top \vec{u}$ ▷ Change the external input functions to be in V-coordinates
3: **for** $i = n$ down to 1 **do** ▷ Iterate up from the bottom row
4: $\quad \check{u}_i = \clubsuit + \sum_{j=i+1}^{n} \spadesuit$ ▷ Make the effective input for this level
5: $\quad \widetilde{x}_i = \texttt{ScalarSolve}(\diamondsuit, \widetilde{x}_{0,i}, \check{u}_i)$ ▷ Solve this level's scalar differential equation
6: **end for**
7: $\vec{x}(t) = \heartsuit \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_n \end{bmatrix}(t)$ ▷ Change back into original coordinates

(a) **Give the expression for $\heartsuit$ on line 7 of the algorithm above.** (i.e., how do you get from $\vec{\widetilde{x}}(t)$ to $\vec{x}(t)$?)

Solution: Since $\vec{\widetilde{x}}_0 = V^\top \vec{x}_0$ we know we are changing to V-basis. So, the implicit change of variable that we are doing is $\vec{\widetilde{x}} = V^\top \vec{x}$, this means that to come back, $\vec{x} = V\vec{\widetilde{x}}$ (since V is orthonormal, $(V^\top)^{-1} = V$). Thus, $\heartsuit = V$.

(b) **Give the expression for $\diamondsuit$ on line 5 of the algorithm above.** (i.e., what are the λ arguments to `ScalarSolve`, equation (2), for the i^{th} iteration of the for-loop?)

(HINT: Convert the differential equation to be in terms of R instead of A. It may be helpful to start with $i = n$ and develop a general form for the i^{th} row.)

Solution: We begin by taking our vector differential equation and substituting in our upper triangularization:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t) \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = VRV^\top\vec{x}(t) + \vec{u}(t) \tag{51}$$

Multiplying both sides by $V^\top$ and using the fact that $V^\top V = I$

$$V^\top \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = RV^\top\vec{x}(t) + V^\top\vec{u}(t) \tag{52}$$

Now, we perform change of variables, $\vec{\widetilde{x}} = V^\top \vec{x}$ and $\vec{\widetilde{u}} = V^\top \vec{u}$ so we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = R\vec{\widetilde{x}}(t) + \vec{\widetilde{u}}(t) \tag{53}$$

Thus, the i^{th} equation in this system is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_i(t) = r_i^\top \vec{\widetilde{x}}(t) + \widetilde{u}_i(t) \tag{54}$$

Using, $\vec{r}_i^\top = \begin{bmatrix} \underbrace{0, \ldots, 0}_{i-1 \text{ times}}, \lambda_i, r_{i,i+1}, r_{i,i+2}, \ldots, r_{i,n} \end{bmatrix}$ we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_i(t) = \lambda_i\widetilde{x}_i(t) + r_{i,i+1}\widetilde{x}_{i+1}(t) + r_{i,i+2}\widetilde{x}_{i+2}(t) + \cdots + r_{i,n}\widetilde{x}_n(t) + \widetilde{u}_i(t) \tag{55}$$

$$= \lambda_i\widetilde{x}_i(t) + \widetilde{u}_i(t) + \sum_{j=i+1}^{n} r_{i,j}\widetilde{x}_j(t) \tag{56}$$

Defining $\check{u}_i(t) = \widetilde{u}_i(t) + \sum_{j=i+1}^{n} r_{i,j}\widetilde{x}_j(t)$ we can write eq. (56) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_i(t) = \lambda_i\widetilde{x}_i(t) + \check{u}_i(t) \tag{57}$$

Here, we can see that when solving the scalar differential equation for the ith row, the scaling term is λ_i: $\diamondsuit = \lambda_i$.

(c) **Give the expression for ♣ on line 4 of the algorithm above.**

Solution: Since, from above, $\check{u}_i(t) = \widetilde{u}_i(t) + \sum_{j=i+1}^{n} r_{i,j}\widetilde{x}_j(t)$ we can see that the $\widetilde{u}_i$ is the input term that does not depend on the inner sum. From this we conclude that $\clubsuit = \widetilde{u}_i$.

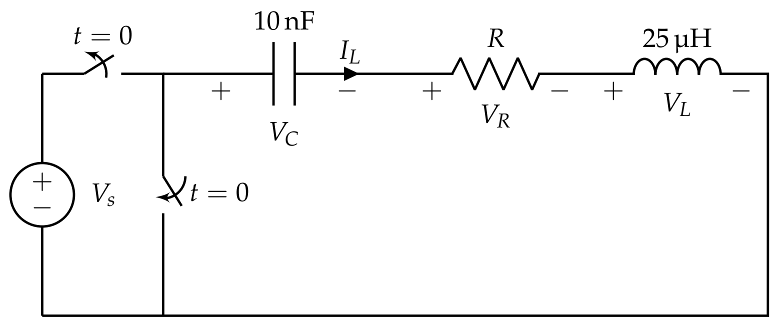
(d) **Give the expression for ♠ on line 4 of the algorithm above.**

Solution: Since, from above, $\check{u}_i(t) = \widetilde{u}_i(t) + \sum_{j=i+1}^{n} r_{i,j}\widetilde{x}_j(t)$ and so we know what is inside the inner sum: $\spadesuit = r_{i,j}\widetilde{x}_j$.

Congratulations! You now know how to systematically solve any system of differential equations with constant coefficients, as long as you know how to solve the scalar case with inputs. The same argument style applies for recurrence relations. The only gap that remains is the assumption that all the eigenvalues are real, but once you understand orthogonality for complex vectors, you can also update your understanding of upper-triangularization to allow for complex matrices as well.

5. RLC Responses: Critically Damped Case

Recall the series RLC circuit we worked on in Homework 4. We're going to re-visit this problem for a special case: the critically-damped case. (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, you may use a calculator that can handle matrices (Matlab, Mathematica, Wolfram Alpha) or the attached `RLC_Calc.ipynb` Jupyter Notebook for numerical calculations.

Recall from Homework 4 that we were able to represent this series RLC circuit as a system of equations in vector/matrix form. We let the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} I_L(t) \\ V_C(t) \end{bmatrix}$ and were able to write the system in the form $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A:

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{58}$$

where

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{59}$$

Further, we showed that the two eigenvalues of A are:

$$\lambda_1 = -\frac{1}{2}\frac{R}{L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{60}$$

$$\lambda_2 = -\frac{1}{2}\frac{R}{L} - \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}} \tag{61}$$

(a) **For what value of R are the two eigenvalues of A going to be identical?** We will refer to this value as R_{crit} later on.

Solution: If the terms under the square root, i.e., the discriminant of the quadratic formula, is 0, then the eigenvalues have a single solution. More concretely,

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0 \tag{62}$$

$$\frac{R^2}{L^2} = \frac{4}{LC} \tag{63}$$

$$R = \pm\sqrt{\frac{4L}{C}} \tag{64}$$

Since R is a physical resistor, R must be non-negative. Therefore:

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{25 \times 10^{-6}}{10 \times 10^{-9}}} = 100\,\Omega \tag{65}$$

(b) Using the given values for the capacitor and the inductor, as well as R_{crit} you found in the previous part, **find the eigenvalues and eigenspaces of A. What are the dimensions of the corresponding eigenspaces?** (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)

It may be easier to plug in numbers into the matrix first.

Solution: Our system's matrix becomes,

$$A = \begin{bmatrix} -4 \times 10^6 & -4 \times 10^4 \\ 10^8 & 0 \end{bmatrix} \tag{66}$$

Our eigenvalues λ_1 and λ_2 are identical, i.e.

$$\lambda_1 = \lambda_2 = \lambda = -\frac{R}{2L} = -2 \times 10^6 \tag{67}$$

Since the two eigenvalues are identical, we expect the corresponding eigenvectors $\vec{v}_1$ to be equal (or some scaled version of) $\vec{v}_2$. We can solve for $\vec{v}_1$ and $\vec{v}_2$ by finding the nullspace of $A - \lambda I$. Let's call the solution to this some generic eigenvector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$\begin{bmatrix} -2 \times 10^6 & -4 \times 10^4 \\ 10^8 & 2 \times 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{68}$$

Notice that the second column of the matrix above is $-50\times$ the first column. Therefore, $\vec{v}_1 = \vec{v}_2 = \begin{bmatrix} 1 \\ -50 \end{bmatrix}$. Furthermore, the nullspace is given by any multiple of the eigenvector, $\alpha \begin{bmatrix} 1 \\ -50 \end{bmatrix}$; we have a single dimensional nullspace and so this matrix A just has a one-dimensional eigenspace, even though the matrix A is 2×2.

You should have found above that the dimension of the eigenspace is 1, i.e. the eigenvectors are linearly dependent. Herein lies the problem. In the overdamped, underdamped, and undamped RLC cases we saw previously in Homework 4, we were able to solve for a system of equation by conducting a coordinate transformation using the eigenvectors of A. In other words, we were able to construct a matrix $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ where $\vec{v}_{\lambda_1}$ and $\vec{v}_{\lambda_2}$ are the linearly independent eigenvalues of A. We could then diagonalize the matrix A as $\widetilde{A} = V^{-1}AV = \Lambda$ and solve for the transformed system. The problem we have now is that the eigenvectors $\vec{v}_{\lambda_1}$ and $\vec{v}_{\lambda_2}$ of A are not linearly dependent, so $V = \begin{bmatrix} \vec{v}_{\lambda_1} & \vec{v}_{\lambda_2} \end{bmatrix}$ is not invertible. Therefore, A is not diagonalizable.

The downside here is we can't diagonalize A to convert our matrix-vector equation to a system of uncoupled scalar equations, which we could solve for independently for each of our state variables. However, we know that even if we can't diagonalize A, we can still upper triangularize it. This yields a system of chained scalar equations, which we can use to solve for our state variables via back-substitution.

To triangularize, we first must find an appropriate change of basis matrix $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}$. This lets us calculate our triangular matrix $T = U^{-1}AU$. If U is an orthonormal matrix (orthogonal column vectors which are normalized), then we can exploit the fact that $U^{-1} = U^T$ and calculate $T = U^T AU$.

(c) There are multiple ways to find an upper triangular matrix of A, and it is not unique. Regardless of your previous result, assume the system matrix is:

$$A = \begin{bmatrix} -4\times 10^6 & -4\times 10^4 \\ 10^8 & 0 \end{bmatrix} \tag{69}$$

If you use the Real Schur Decomposition algorithm from Note 15, you would find an upper triangular matrix T and the associated basis U for the system matrix A given above. For brevity, we will provide you with the basis U:

$$U = \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix} \tag{70}$$

Note that U is an orthonormal matrix. **Find the associated triangular matrix T.** You may use your favorite matrix calculator, e.g. Python, Jupyter notebook, MATLAB, Mathematica, Wolfram Alpha, etc.

Solution: The triangular matrix T is:

$$T = U^{-1}AU \tag{71}$$

Since U is an orthonormal matrix:

$$\begin{aligned} T &= U^T AU & (72)\\ &= \left(\frac{1}{\sqrt{2501}}\begin{bmatrix} 1 & -50 \\ 50 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} -4\times 10^6 & -4\times 10^4 \\ 10^8 & 0 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2501}}\begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix}\right) & (73)\\ &= \frac{1}{2501}\begin{bmatrix} 1 & -50 \\ 50 & 1 \end{bmatrix}\begin{bmatrix} -4\times 10^6 & -4\times 10^4 \\ 10^8 & 0 \end{bmatrix}\begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix} & (74)\\ &= \begin{bmatrix} -2\times 10^6 & -1.0004\times 10^8 \\ 0 & -2\times 10^6 \end{bmatrix} & (75) \end{aligned}$$

It is ok if you rounded off -1.0004×10^8 to -1×10^8.

NOTE: The rest of the solution below is purely for informational purposes as a worked numerical example of the Real Schur Decomposition algorithm for this system. It is not asked for — and not required — in the student solution. However, the Real Schur Decomposition algorithm is within the scope of the course.

If we follow the Real Schur Decomposition algorithm from Note 15, we can find the upper triangular matrix T and the associated basis U for the system matrix A. When extending our basis to $\mathbb{R}^2$ for the Gram-Schmidt procedure, we choose $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (i.e. the columns of the 2×2 identity matrix).

For accuracy, we carry through any square roots until the end and be as precise with decimal points as possible.

We have shown already that our system matrix $A = \begin{bmatrix} -4\times 10^6 & -4\times 10^4 \\ 10^8 & 0 \end{bmatrix}$ has one independent eigenvector $v_\lambda = \begin{bmatrix} 1 \\ -50 \end{bmatrix}$ with eigenvalue $\lambda = -2 \times 10^6$. Following the Real Schur Decomposition algorithm from Note 15, this means:

$$\vec{q}_1 = \begin{bmatrix} 1 \\ -50 \end{bmatrix} \tag{76}$$

$$\lambda_1 = -2 \times 10^6 \tag{77}$$

Continuing the algorithm, we aim to extend $\vec{q}_1$ to a basis Q of $\mathbb{R}^2$ using the vectors $\vec{e}_1$ and $\vec{e}_2$ given in the problem. We must construct the matrix Q using the Gram-Schmidt procedure from Note 13.

Gram-Schmidt($\vec{a}_1, \vec{a}_2, \vec{a}_3$) = Gram-Schmidt($\vec{q}_1, \vec{e}_1, \vec{e}_2$) = Gram-Schmidt($\begin{bmatrix} 1 \\ -50 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$):

$$\vec{z}_1 = \vec{a}_1 = \begin{bmatrix} 1 \\ -50 \end{bmatrix} \tag{78}$$

$$\to \vec{p}_1 = \frac{\vec{z}_1}{||\vec{z}_1||} = \frac{1}{\sqrt{1^2 + (-50)^2}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} = \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \tag{79}$$

$$\vec{z}_2 = \vec{a}_2 - \langle \vec{a}_2, \vec{p}_1 \rangle \vec{p}_1 \tag{80}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \right) \tag{81}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2501} (1) \begin{bmatrix} 1 \\ -50 \end{bmatrix} \tag{82}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2501} \\ \frac{-50}{2501} \end{bmatrix} \tag{83}$$

$$= \begin{bmatrix} \frac{2500}{2501} \\ \frac{50}{2501} \end{bmatrix} \tag{84}$$

$$\to \vec{p}_2 = \frac{\vec{z}_2}{||\vec{z}_2||} = \frac{1}{\sqrt{\left(\frac{2500}{2501}\right)^2 + \left(\frac{50}{2501}\right)^2}} \begin{bmatrix} \frac{2500}{2501} \\ \frac{50}{2501} \end{bmatrix} = \frac{2501}{50\sqrt{50^2 + 1^2}} \begin{bmatrix} \frac{2500}{2501} \\ \frac{50}{2501} \end{bmatrix} = \frac{1}{\sqrt{2501}} \begin{bmatrix} 50 \\ 1 \end{bmatrix} \tag{85}$$

$$\vec{z}_3 = \vec{a}_3 - (\langle \vec{a}_3, \vec{p}_2 \rangle \vec{p}_2 + \langle \vec{a}_3, \vec{p}_1 \rangle \vec{p}_1) \tag{86}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left[\left(\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2501}} \begin{bmatrix} 50 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2501}} \begin{bmatrix} 50 \\ 1 \end{bmatrix} \right) + \left(\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \right) \right] \tag{87}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\left(\frac{1}{2501} \right) \begin{bmatrix} 50 \\ 1 \end{bmatrix} + \left(\frac{-50}{2501} \right) \begin{bmatrix} 1 \\ -50 \end{bmatrix} \right) \tag{88}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} \frac{50}{2501} \\ \frac{1}{2501} \end{bmatrix} + \begin{bmatrix} \frac{-50}{2501} \\ \frac{2500}{2501} \end{bmatrix} \right) \tag{89}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{90}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{91}$$

$$\to \vec{p}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{92}$$

As expected, Gram-Schmit produces 2 non-zero vectors, $\vec{p}_1$ and $\vec{p}_2$, and a zero vector, $\vec{p}_3$. We discard the zero vectors and return:

$$\vec{q}_1 := \vec{p}_1 = \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 \\ -50 \end{bmatrix} \tag{93}$$

$$\vec{q}_2 := \vec{p}_2 = \frac{1}{\sqrt{2501}} \begin{bmatrix} 50 \\ 1 \end{bmatrix} \tag{94}$$

We then return to our Real Schur Decompositon to construct our basis Q:

$$Q = \begin{bmatrix} \vec{q}_1 & \widetilde{Q} \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} = \frac{1}{\sqrt{2501}} \begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix} \tag{95}$$

and can calculate $Q^T AQ$:

$$Q^T AQ = \left(\frac{1}{\sqrt{2501}}\begin{bmatrix}1 & 50\\ -50 & 1\end{bmatrix}^T\right)\begin{bmatrix}-4\times 10^6 & -4\times 10^4\\ 10^8 & 0\end{bmatrix}\left(\frac{1}{\sqrt{2501}}\begin{bmatrix}1 & 50\\ -50 & 1\end{bmatrix}\right) \tag{96}$$

$$= \frac{1}{2501}\begin{bmatrix}1 & -50\\ 50 & 1\end{bmatrix}\begin{bmatrix}-4\times 10^6 & -4\times 10^4\\ 10^8 & 0\end{bmatrix}\begin{bmatrix}1 & 50\\ -50 & 1\end{bmatrix} \tag{97}$$

$$= \begin{bmatrix}-2\times 10^6 & -1.0004\times 10^8\\ 0 & -2\times 10^6\end{bmatrix} \tag{98}$$

$$:= \begin{bmatrix}\lambda_1 & \vec{\tilde{a}}_{12}^T\\ \vec{0}_1 & \widetilde{A}_{22}\end{bmatrix} \tag{99}$$

We numerically solved the matrix multiplication using our favorite matrix calculator. First, we note that the top left entry is our $\lambda = -2\times 10^6$ of our system as was found previously. Next, we note $\vec{\tilde{a}}_{12}^T$ is our top right entry -1.0004×10^8 (try to keep the decimal points, but its fine if you didn't). Third, the bottom left entry is the (1-D) zero vector, as expected. Fourth, the bottom right entry is defined in the Real Schur Decomposition algorithm as the sub-matrix $\widetilde{A}_{22}$. For this case, this sub-matrix is simply a 1×1 matrix, i.e. $\widetilde{A}_{22} = \begin{bmatrix}-2\times 10^6\end{bmatrix}$.

We now recursively call the Real Schur Decomposition algorithm on this sub-matrix $\widetilde{A}_{22}$. Since it is 1×1, we simply return:

$$P = 1 \tag{100}$$

$$\widetilde{T} = \widetilde{A}_{22} = \begin{bmatrix}-2\times 10^6\end{bmatrix} \tag{101}$$

Returning from our recursive sub-call, we construct the change of basis matrix $U := \begin{bmatrix}\vec{q}_1 & \widetilde{Q}P\end{bmatrix}$. Recall that we had constructed $Q = \begin{bmatrix}\vec{q}_1 & \widetilde{Q}\end{bmatrix} = \begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix}$, hence $\widetilde{Q} = \begin{bmatrix}\vec{q}_2\end{bmatrix}$. Since $P = \begin{bmatrix}1\end{bmatrix}$, this means $\widetilde{Q}P = \vec{q}_2$ and hence:

$$U = \begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix} = Q = \frac{1}{\sqrt{2501}}\begin{bmatrix}1 & 50\\ -50 & 1\end{bmatrix} \tag{102}$$

Likewise, we can calculate our final triangular matrix T:

$$T := \begin{bmatrix}\lambda_1 & \vec{\tilde{a}}_{12}^T P\\ \vec{0}_1 & \widetilde{T}\end{bmatrix} = \begin{bmatrix}-2\times 10^6 & -1.0004\times 10^8\\ 0 & -2\times 10^6\end{bmatrix} \tag{103}$$

(d) We have solved for a coordinate system U which triangularizes our system matrix A to the T we found. We apply a change of basis to define $\vec{\tilde{x}}$ in the transformed coordinates such that $\vec{x}(t) = U\vec{\tilde{x}}(t)$. **Starting from $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t)$, show that the differential equation in the transformed coordinate system is $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = T\vec{\tilde{x}}(t)$**

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{104}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(U\vec{\tilde{x}}(t)\right) = A\left(U\vec{\tilde{x}}(t)\right) \tag{105}$$

$$U\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = (AU)\,\vec{\tilde{x}}(t) \tag{106}$$

$$U^{-1}U\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = U^{-1}(AU)\,\vec{\tilde{x}}(t) \tag{107}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = \left(U^{-1}AU\right)\vec{\tilde{x}}(t) \tag{108}$$

Since U is an orthonormal matrix, $U^{-1} = U^T$, and hence:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \left(U^T A U\right)\vec{\widetilde{x}}(t) \tag{109}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = T\vec{\widetilde{x}}(t) \tag{110}$$

Although not required, we can write this numerically as:

$$\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t)\end{bmatrix} = \begin{bmatrix}-2\times10^6 & -1.0004\times10^8\\ 0 & -2\times10^6\end{bmatrix}\begin{bmatrix}\widetilde{x}_1(t)\\ \widetilde{x}_2(t)\end{bmatrix} \tag{111}$$

(e) Notice that the second differential equation for $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t)$ in the above coordinate system only depends on $\widetilde{x}_2(t)$ itself. There is no cross-term dependence. This happened because we transformed into a coordinate system which triangularizes A. **Compute the initial condition for $\widetilde{x}_2(0)$ and write out the solution to this scalar differential equation for $\widetilde{x}_2(t)$ for $t \geq 0$.** Assume that $V_s = 1\,\mathrm{V}$. Recall that, in our original system, $\vec{x}(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}I_L(t)\\ V_C(t)\end{bmatrix}$.

Solution: First let's compute $\widetilde{x}_2(0)$, noting that our basis matrix U is an orthonormal matrix and hence $U^{-1} = U^T$:

$$\vec{\widetilde{x}} = U^{-1}\vec{x} = U^T\vec{x} \tag{112}$$

$$\begin{bmatrix}\widetilde{x}_1(0)\\ \widetilde{x}_2(0)\end{bmatrix} = \frac{1}{\sqrt{2501}}\begin{bmatrix}1 & -50\\ 50 & 1\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} \tag{113}$$

$$= \frac{1}{\sqrt{2501}}\begin{bmatrix}1 & -50\\ 50 & 1\end{bmatrix}\begin{bmatrix}I_L(0)\\ V_C(0)\end{bmatrix} \tag{114}$$

$$= \frac{1}{\sqrt{2501}}\begin{bmatrix}1 & -50\\ 50 & 1\end{bmatrix}\begin{bmatrix}0\\ V_S\end{bmatrix} \tag{115}$$

$$= \frac{1}{\sqrt{2501}}\begin{bmatrix}-50V_S\\ V_S\end{bmatrix} \tag{116}$$

$$= \frac{1}{\sqrt{2501}}\begin{bmatrix}-50\\ 1\end{bmatrix} \tag{117}$$

$$\widetilde{x}_1(0) = \frac{-50}{\sqrt{2501}} \tag{118}$$

$$\widetilde{x}_2(0) = \frac{1}{\sqrt{2501}} \tag{119}$$

Solving the following differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t) = -(2\times10^6)\widetilde{x}_2$$
$$\widetilde{x}_2(t) = K_2 \mathrm{e}^{-(2\times10^6)t}.$$

Substituting for the initial condition, we get $K_2 = \frac{1}{\sqrt{2501}}$, i.e. $\widetilde{x}_2(t) = \frac{1}{\sqrt{2501}}\mathrm{e}^{-(2\times10^6)t}$.

(f) With an explicit solution to $\widetilde{x}_2(t)$ in hand, use this to back-substitute and **write out the resulting scalar differential equation for $\widetilde{x}_1(t)$.** This should effectively have a time-dependent input (i.e. $\widetilde{x}_2(t)$) in it. **Also compute the initial condition for $\widetilde{x}_1(0)$** (which you may have already solved for in the previous part).

Solution: We already calculated the initial condition $\widetilde{x}_1(0) = \frac{-50}{\sqrt{2501}}$ in (118).

The differential equation for $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1(t) = \left(-2\times10^6\right)\widetilde{x}_1(t) + \left(-1.0004\times10^8\right)\widetilde{x}_2(t), \tag{120}$$

which is just the top row of the matrix equation $\frac{d}{dt}\widetilde{x}(t) = T\widetilde{x}(t)$ from (111). Substituting in the solution we found for $\widetilde{x}_2(t)$ gives

$$\frac{d}{dt}\widetilde{x}_1(t) = -2\times 10^6\widetilde{x}_1(t) + \frac{-1.0004\times 10^8}{\sqrt{2501}}e^{-(2\times 10^6)t}. \tag{121}$$

Just like we expected, this is a scalar differential equation with an input.

(g) The differential equation you found should look like a standard form seen in a previous homework with a non-zero input. The input here happens to be exponential. **Solve the above scalar differential equation for $\widetilde{x}_1(t)$ over $t \geq 0$.** *(HINT: See Homework 2, Question 4.)*

Solution: Recall from the earlier homework, we proved that the differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t), \tag{122}$$

with initial value $x(0) = x_0$, has the unique solution

$$x(t) = e^{\lambda t}x_0 + \int_0^t e^{\lambda(t-\tau)}u(\tau)\,d\tau\,. \tag{123}$$

Now, the differential equation we found for $\widetilde{x}_1(t)$ in the previous part has this form, with $u(t) = ae^{\lambda t}$, $\lambda = -2\times 10^6$, $a = \frac{-1.0004\times 10^8}{\sqrt{2501}}$. Also note that the initial condition is $x_0 = \widetilde{x}_1(0) = \frac{-50}{\sqrt{2501}}$ from (118). Solving for $\widetilde{x}_1(t)$ using this specific $u(t)$,

$$\begin{aligned}
\widetilde{x}_1(t) &= e^{\lambda t}x_0 + \int_0^t e^{\lambda t-\lambda\tau}(ae^{\lambda\tau})\,d\tau && (124)\\
&= e^{\lambda t}x_0 + e^{\lambda t}\int_0^t a\,d\tau && (125)\\
&= e^{\lambda t}x_0 + ate^{\lambda t} && (126)\\
&= e^{\lambda t}(x_0 + at) && (127)\\
&= x_0 e\lambda t + ate^{\lambda t} && (128)\\
&= -\frac{50}{\sqrt{2501}}e^{-(2\times 10^6)t} - \left(\frac{1.0004\times 10^8}{\sqrt{2501}}\right)te^{-(2\times 10^6)t}. && (129)
\end{aligned}$$

(h) **Find $x_1(t)$ and $x_2(t)$ for $t \geq 0$ based on the answers to the previous three parts.**

Solution: Now that we have $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$, all we need to do to find $x_1(t)$ and $x_2(t)$ is to reverse the coordinate change we made. In other words, we can find $x(t)$ as

$$x(t) = U\widetilde{x}(t). \tag{130}$$

This gives

$$\begin{aligned}
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \frac{1}{\sqrt{2501}}\begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix}\begin{bmatrix} \widetilde{x}_1(t) \\ \widetilde{x}_2(t) \end{bmatrix} && (131)\\
&= \frac{1}{\sqrt{2501}}\begin{bmatrix} 1 & 50 \\ -50 & 1 \end{bmatrix}\begin{bmatrix} -\frac{50}{\sqrt{2501}}e^{-(2\times 10^6)t} - \left(\frac{1.0004\times 10^8}{\sqrt{2501}}\right)te^{-(2\times 10^6)t} \\ \frac{1}{\sqrt{2501}}e^{-(2\times 10^6)t} \end{bmatrix} && (132)\\
&= \frac{1}{2501}\begin{bmatrix} -(1.0004\times 10^8)te^{-(2\times 10^6)t} \\ (2501)e^{-(2\times 10^6)t} + (50\times 1.0004\times 10^8)te^{-(2\times 10^6)t} \end{bmatrix} && (133)\\
&= \begin{bmatrix} -(4\times 10^4)te^{-(2\times 10^6)t} \\ (1 + (2\times 10^6)t)e^{-(2\times 10^6)t} \end{bmatrix} && (134)
\end{aligned}$$

This solution form is what makes the critically-damped case unique: it has the form $te^{\lambda t}$.

(i) In the `RLCSliders.ipynb` Jupyter notebook, move the resistance slider to your R_{crit} value. Next, move the slider so that the resistance is slightly lower — yielding an underdamped system — and slightly higher – yielding an overdamped system (both of which you already solved for in Homework 4). **Qualitatively describe the settling response and the eigenvalues for the underdamped, critically damped, and overdamped cases. Is there any ringing in the time response? Are the eigenvalues real, complex conjugates, identical?** For the underdamped and overdamped cases, you can use any resistance value. For the critically damped case, pick your resistance precisely.

Solution: The main takeaway here is that the critically damped case acts as the boundary between the time-domain solution having ringing (and overshoots) in the underdamped case vs. not in the overdamped case. In terms of eigenvalues: the underdamped case has purely imaginary complex-conjugate eigenvalues; the critically damped case has identical real eigenvalues; and the overdamped case has separate real eigenvalues.

As an aside, the critically damped case is the fastest settling time you can get from the system without overshooting. The underdamped case can achieve faster settling, but your system must be able to tolerate the ringing. In more advanced circuit courses, you'll look into settling time within a final settling error (e.g. $\pm 5\%$ of final value) and how to design your system for an appropriate level of damping to achieve the fastest acceptable response. This is important when pushing for performance in high-speed ADCs, digital circuits, and the like.

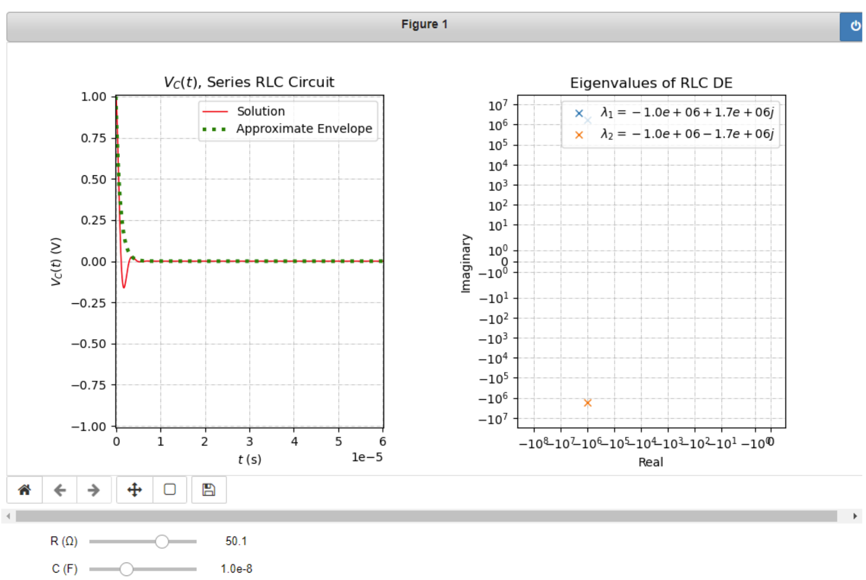


Figure 1: RLC underdamped case: Ringing in transient solution, purely imaginary complex-conjugate eigenvalues.

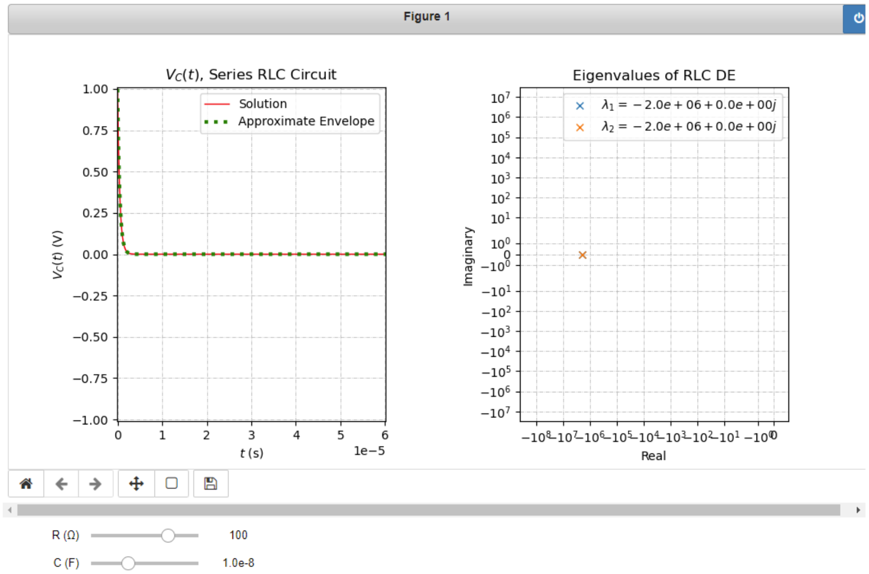


Figure 2: RLC critically damped case: No ringing, identical real eigenvalues.

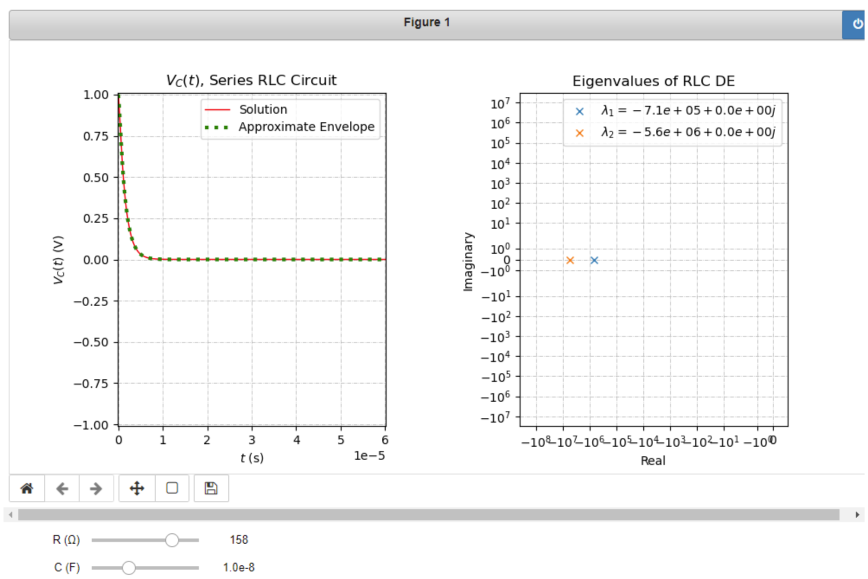


Figure 3: RLC overdamped case: No ringing, different real eigenvalues.

6. (OPTIONAL) Correctness of the Gram-Schmidt Algorithm

NOTE: This problem is optional because we wanted to make the homework shorter. However, all the concepts covered are in scope, and are fair game for future content, including the final exam.

Suppose we take a list of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ and run the following Gram-Schmidt algorithm on it to perform orthonormalization. It produces the vectors $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$.

```
1: for i = 1 up to n do                                  ▷ Iterate through the vectors
2:     r_i = a_i − Σ_{j<i} q_j (q_j^T a_i)                ▷ Find the amount of a_i that remains after we project
3:     if r_i = 0 then
4:         q_i = 0
5:     else
6:         q_i = r_i / ||r_i||                           ▷ Normalize the vector.
7:     end if
8: end for
```

In this problem, we prove the correctness of the Gram-Schmidt algorithm by showing that the following three properties hold on the vectors output by the algorithm.

1. If $\vec{q}_i \neq \vec{0}$, then $\vec{q}_i^\top \vec{q}_i = \|\vec{q}_i\|^2 = 1$ (i.e. the $\vec{q}_i$ have unit norm whenever they are nonzero).
2. For all $1 \leq \ell \leq n$, $\text{Span}(\{\vec{a}_1, \ldots, \vec{a}_\ell\}) = \text{Span}(\{\vec{q}_1, \ldots, \vec{q}_\ell\})$.
3. For all $i \neq j$, $\vec{q}_i^\top \vec{q}_j = 0$ (i.e. $\vec{q}_i$ and $\vec{q}_j$ are orthogonal).

(a) First, we show that the first property holds by construction from the if/then/else statement in the algorithm. It holds when $\vec{q}_i = \vec{0}$, since the first property has no restrictions on $\vec{q}_i$ if it is the zero vector. **Show that** $\|\vec{q}_i\| = 1$ **if** $\vec{q}_i \neq \vec{0}$.

Solution: When $\vec{r}_i = \vec{0}$, then line 4 of the algorithm will make it terminate with $\vec{q}_i = \vec{0}$. This means that the nonzero case must come from the else statement and so from line 6 of the algorithm, we have $\vec{q}_i = \frac{\vec{r}_i}{\|\vec{r}_i\|}$. We know the norm of $\|\vec{q}_i\| = \vec{q}_i^\top \vec{q}_i$. Expanding this, we see

$$\|\vec{q}_i\|^2 = \vec{q}_i^\top \vec{q}_i = \left(\frac{\vec{r}_i}{\|\vec{r}_i\|}\right)^\top \left(\frac{\vec{r}_i}{\|\vec{r}_i\|}\right). \tag{135}$$

Grouping terms and using the definition of the norm, we get

$$\|\vec{q}_i\|^2 = \frac{1}{\|\vec{r}_i\|^2} \vec{r}_i^\top \vec{r}_i = \frac{1}{\|\vec{r}_i\|^2} \|\vec{r}_i\|^2 = 1. \tag{136}$$

(b) Next, we show the second property by considering each ℓ from 1 to n, and showing the statement that $\text{Span}(\{\vec{a}_1, \ldots, \vec{a}_\ell\}) = \text{Span}(\{\vec{q}_1, \ldots, \vec{q}_\ell\})$. This statement is true when $\ell = 1$ since the algorithm produces $\vec{q}_1$ as a scaled version of $\vec{a}_1$. Now assume that this statement is true for $\ell = k - 1$. Under this assumption, **show that the spans are the same for** $\ell = k$.

This implies that because $\text{Span}(\{\vec{a}_1\}) = \text{Span}(\{\vec{q}_1\})$, then so too is $\text{Span}(\{\vec{a}_1, \vec{a}_2\}) = \text{Span}(\{\vec{q}_1, \vec{q}_2\})$, and so forth, until we get that $\text{Span}(\{\vec{a}_1, \ldots, \vec{a}_n\}) = \text{Span}(\{\vec{q}_1, \ldots, \vec{q}_n\})$.

(HINT: What you need to show is: if there exists $\vec{\alpha} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \neq \vec{0}_k$ *so that* $\vec{y} = \sum_{j=1}^{k} \alpha_j \vec{a}_j$*, then there exists* $\vec{\beta} = \begin{bmatrix} \beta_1 & \cdots & \beta_k \end{bmatrix} \neq \vec{0}_k$ *such that* $\vec{y} = \sum_{j=1}^{k-1} \beta_j \vec{q}_j$ *(this is the forward direction). And vice versa from* $\vec{\beta}$ *to* $\vec{\alpha}$ *(this is the reverse direction).)*

(HINT: To show the forward direction, write $\vec{a}_k$ *in terms of* $\vec{q}_k$ *and earlier* $\vec{q}_j$*. Use the condition for* $\ell = k - 1$ *to show the condition for* $\ell = k$*. Don't forget the case that* $\vec{q}_k = \vec{0}$*. The reverse direction may be approached similarly.)*

Solution: Proof outline: we first show that we can express $\vec{\beta}$ in terms of $\vec{\alpha}$, then vice versa. To do so, we express $\vec{a}_k$ or $\vec{q}_k$ using the algorithm above and then group $\vec{q}_j$ terms.

Assume for all $\ell \leq k-1$, for any $\alpha_1, \ldots, \alpha_{k-1}$ there exists $\beta_1, \ldots, \beta_{k-1}$ such that

$$\sum_{j=1}^{k-1} \beta_j \vec{q}_j = \sum_{j=1}^{k-1} \alpha_j \vec{a}_k \tag{137}$$

Then for $\ell = k$, consider a generic set of α's, $\alpha_1, \ldots, \alpha_{k-1}, \alpha_k$.

$$\sum_{j=1}^{k} \alpha_j \vec{a}_j = \alpha_k \vec{a}_k + \sum_{j=1}^{k-1} \alpha_j \vec{a}_j \tag{138}$$

$$= \alpha_k \left(\|\vec{r}_k\| \vec{q}_k + \sum_{i<k} \vec{q}_i \left(\vec{q}_i^\top \vec{a}_k \right) \right) + \sum_{j=1}^{k-1} \beta_j \vec{q}_j \tag{139}$$

$$= \underbrace{\alpha_k \|\vec{r}_k\|}_{\beta_{k'}} \vec{q}_k + \sum_{j=1}^{k-1} \left(\underbrace{\beta_j + \alpha_k \left(\vec{q}_j^\top \vec{a}_k \right)}_{\beta_{j'}} \right) \vec{q}_j \tag{140}$$

The second equality is from line 2 of the algorithm where it states $\vec{r}_k = \vec{a}_k - \sum_{i<k} \vec{q}_i \left(\vec{q}_i^\top \vec{a}_k \right)$. This can be rearranged as

$$\vec{a}_k = \vec{r}_k + \sum_{i<k} \vec{q}_i \left(\vec{q}_i^\top \vec{a}_k \right) \tag{141}$$

This finishes the first direction of the proof, as we can now write the vector $\vec{y}$ as

$$\vec{y} = \sum_{j=1}^{k} \beta_{j'} \vec{q}_j. \tag{142}$$

Thus, it holds for $\ell = k$. The opposite direction follows similarly, which is shown below.
Once again, we assume that the statement holds for $\ell = k-1$. For any $\beta_1, \ldots, \beta_{k-1}$ there exists $\alpha_1, \ldots, \alpha_{k-1}$ such that

$$\sum_{j=1}^{k-1} \beta_j \vec{q}_j = \sum_{j=1}^{k-1} \alpha_j \vec{a}_j \tag{143}$$

Then for $\ell = k$, consider a generic set of β's, $\beta_1, \ldots, \beta_{k-1}$.

$$\vec{y} = \sum_{j=1}^{k} \beta_j \vec{q}_j = \beta_k \vec{q}_k + \sum_{j=1}^{k-1} \beta_j \vec{q}_j \tag{144}$$

$$= \beta_k \frac{\vec{r}_k}{\|\vec{r}_k\|} + \sum_{j=1}^{k-1} \beta_j \vec{q}_j \tag{145}$$

$$= \frac{\beta_k}{\|\vec{r}_k\|} \left(\vec{a}_k - \sum_{j<k} \vec{q}_j \left(\vec{q}_j^\top \vec{a}_k \right) \right) + \sum_{j=1}^{k-1} \beta_j \vec{q}_j \tag{146}$$

$$= \frac{\beta_k}{\|\vec{r}_k\|} \vec{a}_k + \sum_{j=1}^{k-1} \vec{q}_j \left(\beta_j - \frac{\beta_k}{\|\vec{r}_k\|} \left(\vec{q}_j^\top \vec{a}_k \right) \right). \tag{147}$$

Notice that the second sum is only up through $k-1$ and involves a linear combination of the $\vec{q}_j$. So we can use our assumption: there exists an appropriate set of α that would weight $\vec{a}_j$ to equal that second sum.

$$\vec{y} = \frac{\beta_k}{\|\vec{r}_k\|} \vec{a}_k + \sum_{j=1}^{k-1} \alpha_j \vec{a}_j \tag{148}$$

Here, α_k can be defined to be $\frac{\beta_k}{\|\vec{r}_k\|}$ and we have proved the result for $\ell = k$. Thus, we have the new form of

$$\vec{y} = \sum_{j=1}^{k} \alpha_j \vec{a}_j. \tag{149}$$

(c) Lastly, we establish orthogonality between every pair of vectors in $\{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n\}$. Consider each ℓ from 1 to n. We want to show the statement that for all $j < \ell$, $\vec{q}_j^\top \vec{q}_\ell = 0$. The statement holds for $\ell = 1$ since there are no $j < 1$. Assume that this statement holds for ℓ up to and including $k-1$. That is, we assume that for all $i \leq k-1$, $\vec{q}_j^\top \vec{q}_i = 0$ for all $j < i$.

Under this assumption, **show that for all** $i \leq k$**, that** $\vec{q}_j^\top \vec{q}_i = 0$ **for all** $j < i$**.** This shows that every pair of distinct vectors up to $1, 2, \dots, \ell$ are orthogonal for each ℓ from 1 to n.

(HINT: The cases $i \leq k-1$ are already covered by the assumption. So you can focus on $i = k$. Next, notice that the case $\vec{q}_k = \vec{0}$ is also true, since the inner product of any vector with $\vec{q}_k = \vec{0}$ is $\vec{0}$. So, focus on the case $\vec{q}_k \neq \vec{0}$ and expand what you know about $\vec{q}_k$.)

Solution: The cases $i \leq k-1$ are given by the assumption, as stated in the hint. Then, for all $i \leq k-1$, for all $j < i$, $\vec{q}_j^\top \vec{q}_i = 0$. Remember that $\vec{q}_j^\top \vec{q}_j = 1$. All that remains is to deal with the case of $\vec{q}_k$ itself. We need to verify that it is orthogonal to all the previous $\vec{q}_j$. If $\|\vec{r}_k\| = 0$, then $\vec{q}_k = \vec{0}$ and so it definitely holds since the zero vector is orthogonal to every vector. So now we can assume $\|\vec{r}_k\| > 0$ and check for $i = k$.

For any $j < k$,

$$\vec{q}_j^\top \vec{q}_k = \vec{q}_j^\top \frac{1}{\|\vec{r}_k\|}\left(\vec{a}_k - \sum_{n<k} \vec{q}_n \left(\vec{q}_n^\top \vec{a}_k\right)\right) \tag{150}$$

$$= \frac{1}{\|\vec{r}_k\|}\left(\vec{q}_j^\top \vec{a}_k - \vec{q}_j^\top \sum_{n<k} \vec{q}_n \left(\vec{q}_n^\top \vec{a}_k\right)\right) \tag{151}$$

$$= \frac{1}{\|\vec{r}_k\|}\left(\vec{q}_j^\top \vec{a}_k - \sum_{n<k} \vec{q}_j^\top \vec{q}_n \left(\vec{q}_n^\top \vec{a}_k\right)\right) \tag{152}$$

$$= \frac{1}{\|\vec{r}_k\|}\left(\vec{q}_j^\top \vec{a}_k - \vec{q}_j^\top \vec{a}_k\right) = 0 \tag{153}$$

The final step is a cancellation of the cross terms, since they are all zero. Only when the index $n = j$ in the sum, the $\vec{q}_j^\top \vec{q}_j = 1$ and $\vec{q}_j^\top \vec{a}_k$ survives.

(d) **(OPTIONAL)** The second property can be summarized in matrix form. Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ and $Q = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}$. If A and Q have the same column span then there exists square matrix $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$ such that $A = QU$.

Show that a U can be found that is upper-triangular (i.e. the i-th column $\vec{u}_i$ of U has zero entries in it for every row after the i-th position.)

(HINT: Matrix multiplication tells you that $\vec{a}_i = \sum_{j=1}^{n} \vec{u}_{i_j} \vec{q}_j$. What does the algorithm tell you about this relationship? Can you figure out what $\vec{u}_{i_j}$ should be?)

Solution: This is the kind of question that many students might have gotten stuck on. It is important to know how to start working on such things. As shown repeatedly in lecture and exemplified in discussions, the way to start is small. Our procedure is iterative, and our proofs have been inductive. So we should see what happens and discover the pattern.

While we are looking to find the pattern of interest, we can not worry about the case when vectors are linearly dependent, or zero. We are trying to understand the basic story at this point.

We know that

$$\vec{q}_1 = \frac{1}{\|\vec{a}_1\|}\vec{a}_1 \tag{154}$$

One way to reverse this equation involves solving for an expression of the norm as an inner product of two vectors by multiplying both sides by $\vec{q}_i^\top$.

$$\vec{a}_1 = \vec{q}_1 \|\vec{a}_1\| \tag{155}$$
$$\vec{q}_1^\top \vec{a}_1 = \vec{q}_1^\top \vec{q}_1 \|\vec{a}_1\| \tag{156}$$
$$\vec{q}_1^\top \vec{a}_1 = \|\vec{a}_1\| \tag{157}$$
$$\Rightarrow \vec{a}_1 = \vec{q}_1 \|\vec{a}_1\| = \vec{q}_1 \left(\vec{q}_1^\top \vec{a}_1\right) \tag{158}$$

Here, the second-to-last step comes from the fact that $\vec{q}_1$ is normalized, so $\vec{q}_1^\top \vec{q}_1 = \|\vec{q}_1\|^2 = 1$.
For $\vec{a}_2$,

$$\vec{q}_2 = \frac{\vec{a}_2 - \vec{q}_1(\vec{q}_1^\top \vec{a}_2)}{\|\vec{a}_2 - \vec{q}_1(\vec{q}_1^\top \vec{a}_2)\|} \tag{159}$$
$$\Rightarrow \quad \vec{a}_2 = \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right) + \vec{q}_2 \left\|\vec{a}_2 - \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right)\right\|. \tag{160}$$

To simplify, we again need to find a way to compute the large norm expression. We might suspect that it is equal to $\vec{q}_2^\top \vec{a}_2$, in symmetry with the first term. To check, let us get $\vec{q}_2^\top \vec{a}_2$ on one side and see the other side by left-multiply $\vec{q}_2^\top$ on both sides:

$$\vec{q}_2^\top \vec{a}_2 = \vec{q}_2^\top \left(\vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right) + \vec{q}_2 \left\|\vec{a}_2 - \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right)\right\|\right) \tag{161}$$
$$= \vec{q}_2^\top \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right) + \vec{q}_2^\top \vec{q}_2 \left\|\vec{a}_2 - \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right)\right\|) \tag{162}$$
$$= \underbrace{\left(\vec{q}_2^\top \vec{q}_1\right)}_{=0}\left(\vec{q}_1^\top \vec{a}_2\right) + \underbrace{\left(\vec{q}_2^\top \vec{q}_2\right)}_{=1} \left\|\vec{a}_2 - \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right)\right\| \tag{163}$$
$$= \left\|\vec{a}_2 - \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right)\right\|. \tag{164}$$

The second-to-last line here uses the fact that $\vec{q}_2^\top \vec{q}_1 = 0$ (since these vectors are orthogonal) and $\vec{q}_2$ is normalized, so $\vec{q}_2^\top \vec{q}_2 = \|\vec{q}_2\|^2 = 1$. Thus we can write

$$\vec{a}_2 = \vec{q}_1\left(\vec{q}_1^\top \vec{a}_2\right) + \vec{q}_2\left(\vec{q}_2^\top \vec{a}_2\right). \tag{165}$$

Thus, to get the coordinates for $\vec{a}_2$ in the orthonormal basis given by $\vec{q}_1, \vec{q}_2$, we take the inner products of $\vec{q}_i$ with $\vec{a}_2$.

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2\end{bmatrix} = \begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix}\begin{bmatrix}\vec{q}_1^\top \vec{a}_1 & \vec{q}_1^\top \vec{a}_2 \\ 0 & \vec{q}_2^\top \vec{a}_2\end{bmatrix} \tag{166}$$

Then, for $\vec{a}_3$ we get the same pattern again and can express them all using matrix multiplications:

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix} = \begin{bmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{bmatrix}\begin{bmatrix}\vec{q}_1^\top \vec{a}_1 & \vec{q}_1^\top \vec{a}_2 & \vec{q}_1^\top \vec{a}_3 \\ 0 & \vec{q}_2^\top \vec{a}_2 & \vec{q}_2^\top \vec{a}_3 \\ 0 & 0 & \vec{q}_3^\top \vec{a}_3\end{bmatrix} \tag{167}$$

This pattern allows us to guess that $\vec{u}_{i_j} = \vec{q}_j^\top \vec{a}_i$.
And furthermore, we know why the terms below the diagonal are all zero — they are zero because we don't need those $\vec{q}_j$ to express the relevant $\vec{a}_i$.
Having a clear understanding makes doing the proof much easier. Let us consider the i^{th} column of A, $\vec{a}_i$. We want to understand the weights $\vec{u}_i$ required to satisfy

$$\vec{a}_i = \sum_{j=1}^{n} \vec{u}_{i_j} \vec{q}_j. \tag{168}$$

The relevant term in the algorithm is where $\vec{r}_i$ is being computed. We know $\vec{r}_i = \vec{a}_i - \sum_{j<i} \vec{q}_j \left(\vec{q}_j^\top \vec{a}_i\right)$ and hence $\vec{a}_i = \vec{r}_i + \sum_{j<i} \vec{q}_j \left(\vec{q}_j^\top \vec{a}_i\right)$. If $\vec{r}_i = \vec{0}$, we are already done since we have expressed $\vec{a}_i$ in terms of $\vec{q}_j$ with $j < i$. Otherwise, we know that $\vec{q}_i = \frac{\vec{r}_i}{\|\vec{r}_i\|}$ and hence $\vec{r}_i = \|\vec{r}_i\|\vec{q}_i$. So $\vec{a}_i = \|\vec{r}_i\|\vec{q}_i + \sum_{j<i} \vec{q}_j \left(\vec{q}_j^\top \vec{a}_i\right)$ and in a sense, we are already done since we have expressed $\vec{a}_i$ in terms of $\vec{q}_j$ with $j \leq i$.

However, it is nice to complete the story and notice that $\vec{q}_i^\top \vec{a}_i = \|\vec{r}_i\|\vec{q}_i^\top \vec{q}_i + \sum_{j<i} \vec{q}_i^\top \vec{q}_j \left(\vec{q}_j^\top \vec{a}_i\right) = \|\vec{r}_i\|$ since all the cross terms cancel by the orthogonality proved in the previous part. So indeed $\vec{a}_i = \sum_{j=1}^{i} \vec{q}_j \left(\vec{q}_j^\top \vec{a}_i\right)$. So the pattern we conjectured is actually proved.

7. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

8. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

(a) **What sources (if any) did you use as you worked through the homework?**

(b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

(c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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