

This homework is due on Friday, March 11, 2022, at 11:59PM. Self-grades and HW re-submissions are due on the following Friday, March 18, 2022, at 11:59PM.

1. Reading Lecture Notes

Staying up to date with lectures is an important part of the learning process in this course. Here are links to the notes that you need to read for this week: [Note 11](#), and [Note 12](#).

- (a) **How would you use feedback control to choose the closed-loop eigenvalues of a closed-loop discrete-time system?**
- (b) **What is the matrix test for controllability of a general linear discrete-time system $\vec{x}[i + 1] = A\vec{x}[i] + \vec{b}u[i]$ with a scalar input $u[i]$?**
- (c) **If \vec{b} above were an eigenvector of A , why would this imply that the system is not controllable if the dimension of \vec{x} is larger than 1?**

2. Stability Criterion

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

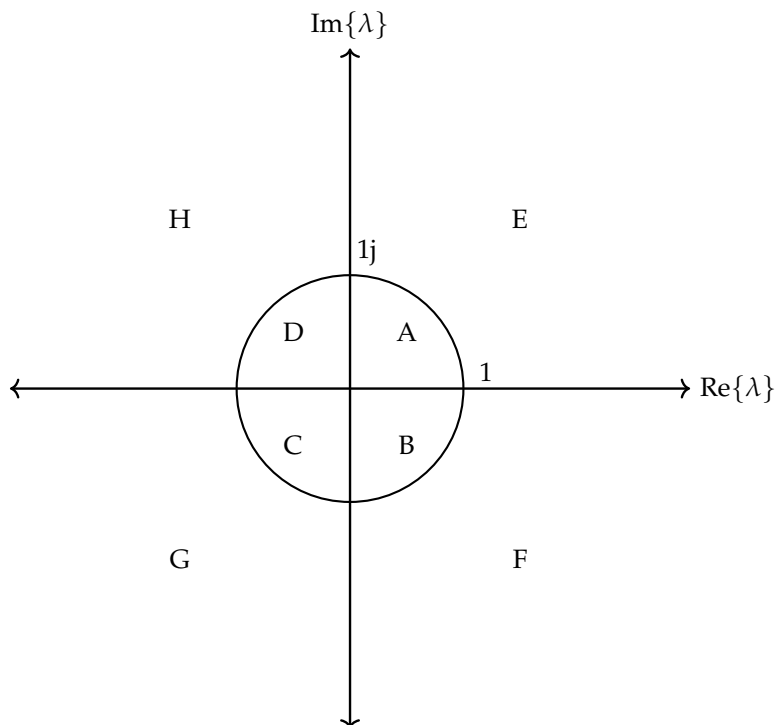


Figure 1: Complex plane divided into regions.

Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$ and the discrete-time system $y[i + 1] = \lambda y[i] + w[i]$. Here $v(t)$ and $w[i]$ are both disturbances to their respective systems.

In which regions can the eigenvalue λ be for the system to be *stable*? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Discrete Time System $y[i]$	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

3. Bounded-Input Bounded-Output (BIBO) Stability

BIBO stability is a system property where bounded inputs lead to bounded outputs. It's important because we want to certify that, provided our system inputs are bounded, the outputs will not “blow up”. In this problem, we gain a better understanding of BIBO stability by considering some simple continuous and discrete systems, and showing whether they are BIBO stable or not.

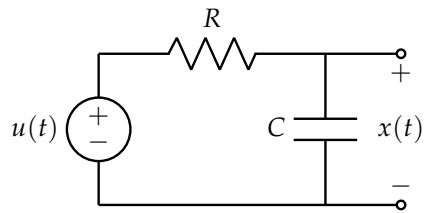
Recall that for the following simple scalar differential equation, we have the corresponding solution:

$$\frac{d}{dt}x(t) = ax(t) + bu(t) \quad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau. \quad (1)$$

And for the following discrete system, we have the corresponding solution:

$$x[i+1] = ax[i] + bu[i] \quad x[i] = a^i x[0] + \sum_{k=0}^{i-1} a^k bu[i-1-k] \quad (2)$$

- (a) Consider the circuit below with $R = 1\Omega$, $C = 0.5F$. Let $x(t)$ be the voltage over the capacitor.



This circuit can be modeled by the differential equation

$$\frac{d}{dt}x(t) = -2x(t) + 2u(t) \quad (3)$$

Intuitively, we know that the voltage on the capacitor can never exceed the (bounded) voltage from the voltage source, so this system is BIBO stable. **Show that this system is BIBO stable, meaning that $x(t)$ remains bounded for all time if the input $u(t)$ is bounded. Equivalently, show that if we assume $|u(t)| < \epsilon$, $\forall t \geq 0$ and $|x(0)| < \epsilon$, then $|x(t)| < M$, $\forall t \geq 0$ for some positive constant M .** Thinking about this helps you understand what bounded-input-bounded-output stability means in a physical circuit.

(HINT: eq. (1) may be useful. You may want to write the expression for $x(t)$ in terms of $u(t)$ and $x(0)$ and then take the norms of both sides to show a bound on $|x(t)|$. Remember that norm in 1D is absolute value. Some helpful formulas are $|ab| = |a||b|$, the triangle inequality $|a+b| \leq |a| + |b|$, and the integral version of the triangle inequality $\left| \int_a^b f(\tau) d\tau \right| \leq \int_a^b |f(\tau)| d\tau$, which just extends the standard triangle inequality to an infinite sum of terms.)

- (b) Assume $x(0) = 0$. **Show that the system eq. (1) is BIBO unstable when $a = j2\pi$ by constructing a bounded input that leads to an unbounded $x(t)$.**

It can be shown that the system eq. (1) is unstable for any purely imaginary a by a similar construction of a bounded input.

- (c) Consider the discrete-time system and its solution in eq. (2). **Show that if $|a| > 1$, then even if $x[0] = 0$, a bounded input can result in an unbounded output, i.e. the system is BIBO unstable.** (HINT: The formula for the sum of a geometric sequence may be helpful.)

- (d) Consider the discrete-time system

$$x[i+1] = -3x[i] + u[i]. \quad (4)$$

Is this system stable or unstable? Give an initial condition $x(0)$ and a sequence of non-zero inputs for which the state $x[i]$ will always stay bounded. (HINT: See if you can find any input pattern that results in an oscillatory behavior.)

4. Eigenvalue Placement through State Feedback

Consider the following discrete-time linear system:

$$\vec{x}[i+1] = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]. \quad (5)$$

In standard language, we have $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the form: $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$.

- Is this system controllable?
- Is this discrete-time linear system stable in open loop (without feedback control)?
- Suppose we use state feedback of the form $u[i] = [f_1 \ f_2] \vec{x}[i] = F\vec{x}[i]$.
Find the appropriate state feedback constants, f_1, f_2 so that the state space representation of the resulting closed-loop system has eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.
- We are now ready to go through some numerical examples to see how state feedback works. Consider the first discrete-time linear system. Enter the matrix A and vector \vec{b} from (a) for the system $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i]$ into the Jupyter notebook "eigenvalue_placement.ipynb" and use the randomly generated $\vec{w}[i]$ as the disturbance introduced into the state equation. Observe how the norm of $\vec{x}[i]$ evolves over time for the given A . **What do you see happening to the norm of the state?**
- Add the feedback computed in part (c) to the system in the notebook and **explain how the norm of the state changes.**
- Now suppose we've got a different system described by the controlled scalar difference equation $z[i+1] = z[i] + 2z[i-1] + u[i]$. To convert this second-order discrete time system to a two-dimensional first-order discrete time system, we will let $\vec{y}[i] = \begin{bmatrix} z[i-1] \\ z[i] \end{bmatrix}$.
Write down the system representation for \vec{y} in the following matrix form:

$$\vec{y}[i+1] = A_y \vec{y}[i] + \vec{b}_y u[i]. \quad (6)$$

Specify the values of the matrix A_y and the vector \vec{b}_y .

- It turns out that the original $\vec{x}[i]$ system can be converted to the $\vec{y}[i]$ system using a change of basis P . Let this coordinate change be written as $\vec{y}[i] = P\vec{x}[i]$. **First express A_y and \vec{b}_y symbolically in terms of A, \vec{b} , and P . Then, confirm numerically that $P = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ is the correct change of basis matrix between the two systems.**
- For the \vec{y} system from part (f), **design a feedback gain matrix $[\bar{f}_1 \ \bar{f}_2]$ to place the closed-loop eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. Additionally, confirm that this matrix is just a change of basis of the gain matrix from part (c), i.e. $[f_1 \ f_2] = [\bar{f}_1 \ \bar{f}_2] P$.**

Note that this means you can solve for the closed-loop gains of your system in any basis, and then transform it to the basis you care about.

5. Open-Loop and Closed-Loop Control

In last week's lab-related System ID problem, we built SIXT33N's motor control circuitry and developed a linear model for the velocity of each wheel. We are one step away from our goal: to have SIXT33N drive in a straight line! We will see how to use the model we developed in the System ID problem to control SIXT33N's trajectory to be a straight line.

More specifically, in this problem, we will explore how to use open-loop and closed-loop control to drive the trajectory of your car in a straight line.

Part 1: Open-Loop Control

An open-loop controller is one in which the input is predetermined using your system model and the goal, and not adjusted at all during operation. To design an open-loop controller for your car, you would set the PWM duty-cycle value of the left and right wheels (inputs $u_L[i]$ and $u_R[i]$) such that the predicted velocity of both wheels is your target wheel velocity (v_t). You can calculate these inputs from the target velocity v_t and the $\theta_L, \theta_R, \beta_L, \beta_R$ values you learned from data. In the System ID problem and lab, we have modeled the velocity of the left and right wheels as

$$v_L[i] = d_L[i+1] - d_L[i] = \theta_L u_L[i] - \beta_L; \quad (7)$$

$$v_R[i] = d_R[i+1] - d_R[i] = \theta_R u_R[i] - \beta_R \quad (8)$$

where $d_{L,R}[i]$ represent the distance traveled by each wheel.

- (a) Find the open-loop control that would give us $v_L[i] = v_R[i] = v_t$. That is, **solve the model (Equations (7) and (8)) for the inputs $u_L[i]$ and $u_R[i]$ that make the velocities $v_L[i] = v_R[i] = v_t$.**

In practice, the $\theta_L, \theta_R, \beta_L, \beta_R$ parameters are learned from noisy data, and so can be wrong. This means that we will calculate the velocities for the two wheels incorrectly. When the velocities of the two wheels disagree, the car will go in a circle instead of a straight line. Thus, to make the car go in a straight line, we need the distances traveled by both wheels to be the same at each timestep.

This prompts us to simplify our model. Instead of having two state variables \vec{v}_L and \vec{v}_R , we can just have a state variable determining how far we are from the desired behavior of going in a line – a state which we will want to drive to 0.

This prompts us to define our state variable δ to be the *difference* in the distance traveled by the left wheel and the right wheel at a given timestep:

$$\delta[i] := d_L[i] - d_R[i] \quad (9)$$

We want to find a scalar discrete-time model for $\delta[i]$ of the form

$$\delta[i+1] = \lambda_{OL} \delta[i] + f(u_L[i], u_R[i]). \quad (10)$$

Here λ_{OL} is a scalar and $f(u_L[i], u_R[i])$ is the control input to the system (as a function of $u_L[i]$ and $u_R[i]$).

- (b) Suppose we apply the open-loop control inputs $u_L[i], u_R[i]$ to the original system. **Using Equations (7) and (8), write $\delta[i+1]$ in terms of $\delta[i]$, in the form of Equation (10). What is the eigenvalue λ_{OL} of the model in Equation (10)? Would the model in Equation (10) be stable with open-loop control if it also had a disturbance term?**

(HINT: For open-loop control, we set the velocities to $v_L[i] = v_R[i] = v_t$. What happens when we substitute that into Equations (7) and (8) and then apply the definition of $\delta[i]$ and $\delta[i+1]$?)

Part 2: Closed-Loop Control

Now, in order to make the car drive straight, we must implement closed-loop control – that is, control inputs that depend on the current state and are calculated dynamically – and use feedback in real time.

- (c) **If we want the car to drive straight starting from some timestep $i_{\text{start}} > 0$, i.e., $v_L[i] = v_R[i]$ for $i \geq i_{\text{start}}$, what condition does this impose on $\delta[i]$ for $i \geq i_{\text{start}}$?**

- (d) **How is the condition you found in the previous part different from the condition:**

$$\delta[i] = 0, \quad i \geq i_{\text{start}}? \quad (11)$$

Assume that $i_{\text{start}} > 0$, and that $d_L[0] = 0, d_R[0] = 0$.

This is a subtlety that is worth noting and often requires one to adjust things in real systems.

- (e) From here, assume that we have reset the distance travelled counters at the beginning of this maneuver so that $\delta[0] = 0$. We will now implement a feedback controller by selecting two dimensionless positive coefficients, f_L and f_R , such that the closed loop system is stable with eigenvalue λ_{CL} . To implement closed-loop feedback control, we want to adjust $v_L[i]$ and $v_R[i]$ at each timestep by an amount that's proportional to $\delta[i]$. Not only do we want our wheel velocities to be some target velocity v_t , we also wish to drive $\delta[i]$ towards zero. This is in order to have the car drive straight along the initial direction it was pointed in when it started moving. If $\delta[i]$ is positive, the left wheel has traveled more distance than the right wheel, so relatively speaking, we can slow down the left wheel and speed up the right wheel to cancel this difference (i.e., drive it to zero) in the next few timesteps. The action of such a control is captured by the following velocities.

$$v_L[i] = v_t - f_L \delta[i]; \quad (12)$$

$$v_R[i] = v_t + f_R \delta[i]. \quad (13)$$

Give expressions for $u_L[i]$ and $u_R[i]$ as a function of $v_t, \delta[i], f_L, f_R$, and our system parameters $\theta_L, \theta_R, \beta_L, \beta_R$, to achieve the velocities above.

- (f) Using the control inputs $u_L[i]$ and $u_R[i]$ found in part (e), **write the closed-loop system equation for $\delta[i+1]$ as a function of $\delta[i]$. What is the closed-loop eigenvalue λ_{CL} for this system in terms of λ_{OL}, f_L , and f_R ?**
- (g) **What is the condition on f_L and f_R for the closed-loop system to be stable in the previous part?**

Stability in this case means that δ is bounded and will not go arbitrarily high. In fact, if our calculated β and θ are perfectly accurate, then $\delta[i] \rightarrow 0$, so the car will (eventually) drive straight!

One question remains – what if our calculated β and θ are *not* perfectly accurate? The answer turns out to be that there is some small steady-state discrepancy that your δ will converge to. You will see how to quantify this in next week's homework.

6. Miscellaneous Practice Problems for Midterm

- (a) You are given the graph in Figure 2. Express the coordinates of vectors \vec{v} and \vec{w} in both Cartesian (x, y) and Polar $(re^{j\theta})$ forms.

You may use the $\text{atan2}()$ or \tan^{-1} function for angle (θ) as necessary.

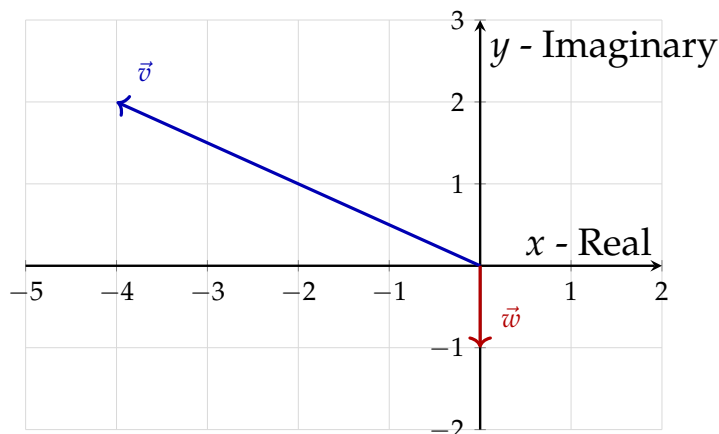


Figure 2: Vectors in the $x - y$ plane

- i. Label \vec{v} with its corresponding Cartesian (x, y) and Polar $(re^{j\theta})$ coordinates, in the given form.
 - ii. Label \vec{w} with its corresponding Cartesian (x, y) and Polar $(re^{j\theta})$ coordinates, in the given form.
- (b) You are given an input voltage signal below:

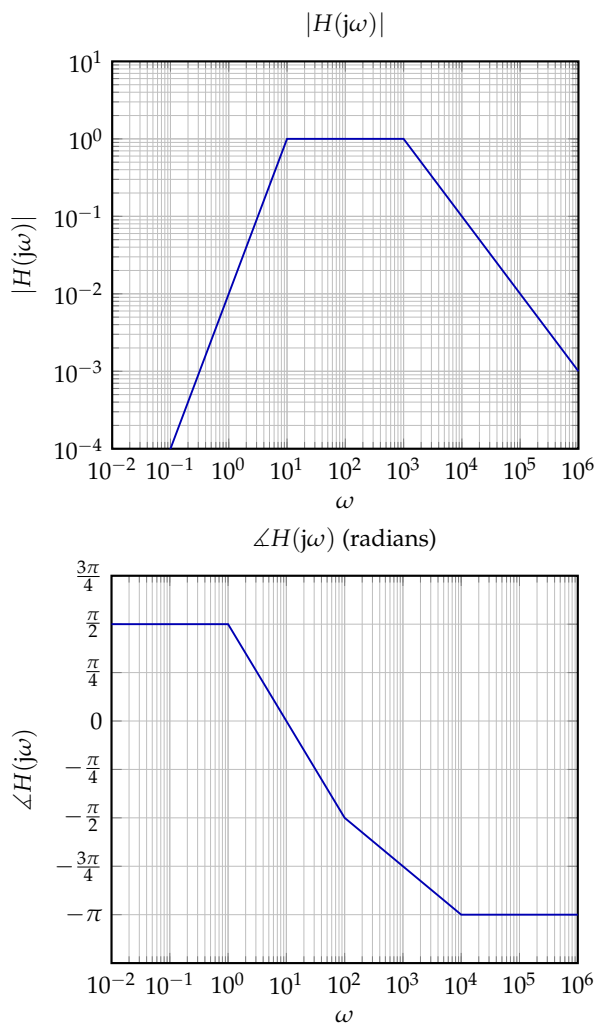
$$v_{\text{in}}(t) = -1.5 \sin\left(\omega t - \frac{\pi}{3}\right). \quad (14)$$

Convert the signal of eq. (14) to its phasor representation. That is, find \tilde{V}_{in} .

- (c) You decided to analyze the transfer function of a band-pass filter, and have generated the following Bode plots for $H(j\omega)$. **If your input voltage signal is**

$$v_{\text{in}}(t) = 4 \sin\left(\omega_s t + \frac{2\pi}{3}\right) \quad (15)$$

where $\omega_s = 1 \times 10^4 \frac{\text{rad}}{\text{s}}$, what is the approximate value of $v_{\text{out}}(t)$ based on the Bode plots? Since the original transfer function is not provided, you cannot numerically compute the exact values of magnitude and phase. Just read the approximate values from the Bode plot.



(d) Assume that the overall transfer function of a new filter, $H(j\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}}$, is given by

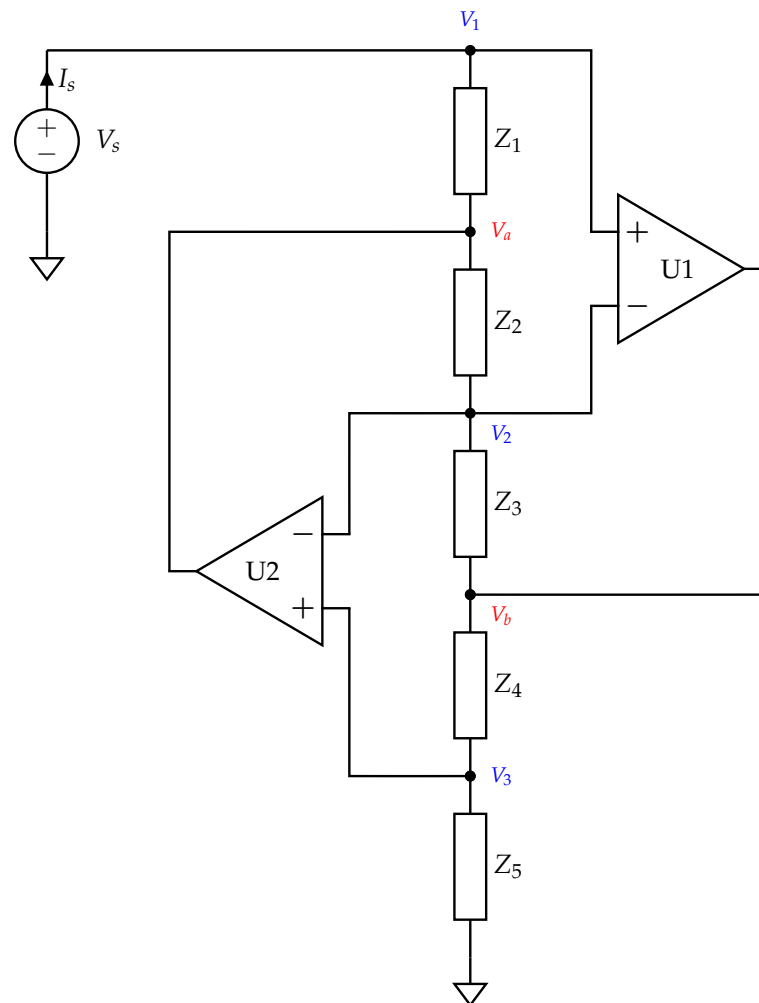
$$H(j\omega) = \left(\frac{1}{1 + j\frac{\omega}{\omega_{c1}}} + \frac{j\frac{\omega}{\omega_{c2}}}{1 + j\frac{\omega}{\omega_{c2}}} \right), \quad (16)$$

where $\omega_{c2} = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function $|H(j\omega)|$ in three regions: frequencies below ω_{c1} , frequencies between ω_{c1} and ω_{c2} , and frequencies above ω_{c2} . Identify the filter type by explaining what it is doing qualitatively (for example, a low-pass filter passes low frequencies but does not pass high frequencies).**

7. Generalized Impedance Converter

“Active inductors” are circuits that make a capacitor act like an inductor with the help of active devices such as transistors and op amps. This can be advantageous when the circuit requires inductors but are significantly larger and non-ideal compared to capacitors and other elements. The tradeoff is that the active devices consume power, but this may be an acceptable design tradeoff. There are many ways of building an active inductor. We are going to analyze one example: the generalized impedance converter (also known as an *Antoniu Gyrator*).

The schematic of a generalized impedance converter is shown below. Consider the circuit in the phasor domain. All the voltages and currents in the problem are phasors and all the Z_i are impedances.



- Treat all the opamps as being in negative feedback and therefore following the Golden Rules. **What are the voltages at V_1 , V_2 , and V_3 in terms of V_s ?**
- Express I_s in terms of V_s , V_a , Z_1 .**
- The input impedance seen by the source looking into the circuit is defined as $Z_{in} = \frac{V_s}{I_s}$. Note that this is true because there are no independent sources in the rest of the circuit (op amps are dependent sources). **Find Z_{in} in terms of Z_1, Z_2, Z_3, Z_4, Z_5 .**
- Assume the following:

$$Z_1 = R_1 \tag{17}$$

$$Z_2 = \frac{1}{j\omega C_2} \quad (18)$$

$$Z_3 = R_3 \quad (19)$$

$$Z_4 = R_4 \quad (20)$$

$$Z_5 = R_5 \quad (21)$$

Evaluate Z_{in} for the above case.

- (e) You should have found that Z_{in} is inductive, i.e. $Z_{in} = j\omega L_{eq}$ where L_{eq} is the equivalent inductance. **What is L_{eq} in terms of $R_1, C_2, R_3, R_4,$ and R_5 ?**

8. (OPTIONAL) Make Your Own Problem.

Write your own problem about content covered in the course thus far, and provide a thorough solution to it.

NOTE: This can be a totally new problem, a modification on an existing problem, or a Jupyter part for a problem that previously didn't have one. Please cite all sources for anything (including course material) that you used as inspiration.

NOTE: High-quality problems may be used as inspiration for the problems we choose to put on future homeworks or exams.

9. Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!

We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- (a) **What sources (if any) did you use as you worked through the homework?**
- (b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)
- (c) **Roughly how many total hours did you work on this homework? Write it down here where you'll need to remember it for the self-grade form.**

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