

Homework 1

This homework is due on Saturday, January 27, 2024, at 11:59PM.

1. Existence and uniqueness of solutions to differential equations

When doing circuits or systems analysis, we sometimes model our system via a differential equation, and would often like to solve it to get the system trajectory. To this end, we would like to verify that a solution to our differential equation exists and is unique, so that our model is physically meaningful. There is a general approach to doing this, which is demonstrated in this problem.

We would like to show that there is a unique function $x: \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$\frac{d}{dt}x(t) = \alpha x(t) \quad (1)$$

$$x(0) = x_0. \quad (2)$$

In order to do this, we will first verify that a solution x_d exists. To show that x_d is the unique solution, we will take an arbitrary solution y and show that $x_d(t) = y(t)$ for every t .

- (a) First, let us show that a solution to our differential equation exists. **Verify that $x_d(t) := x_0 e^{\alpha t}$ satisfies eq. (1) and eq. (2).**
- (b) Now, let us show that our solution is unique. As mentioned before, suppose $y: \mathbb{R} \rightarrow \mathbb{R}$ also satisfies eq. (1) and eq. (2).

We want to show that $y(t) = x_d(t)$ for all t . Our strategy is to show that $\frac{y(t)}{x_d(t)} = 1$ for all t .

However, this particular differential equation poses a problem: if $x_0 = 0$, then $x_d(t) = 0$ for all t , so that the quotient is not well-defined. To patch this method, we would like to avoid using any function with x_0 in the denominator. One way we can do this is consider a modification of the quotient $\frac{y(t)}{x_d(t)} = \frac{y(t)}{x_0 e^{\alpha t}}$; in particular, we consider the function $z(t) := \frac{y(t)}{e^{\alpha t}}$.

Show that $z(t) = x_0$ for all t , and explain why this means that $y(t) = x_d(t)$ for all t .

(HINT: Show first that $z(0) = x_0$ and then that $\frac{d}{dt}z(t) = 0$. Argue that these two facts imply that $z(t) = x_0$ for all t . Then show that this implies $y(t) = x_d(t)$ for all t .)

(HINT: Remember that we said y is any solution to eq. (1) and eq. (2), so we only know these properties of y . If you need something about y to be true, see if you can show it from eq. (1) and eq. (2).)

(HINT: When taking $\frac{d}{dt}z(t)$, remember to use the quotient rule, along with what we know about y .)

2. RC Circuit

The capacitor shown in Figure 1 is charged to a voltage of 50 V prior to $t = 0$.

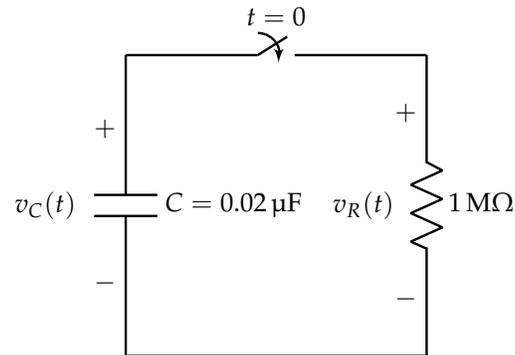


Figure 1: Circuit diagram.

- Find expressions for the voltage across the capacitor $v_C(t)$ and the voltage across the resistor $v_R(t)$.
- Find an expression for the power delivered to the resistor.
- Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered.
- Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

3. RC Circuit with Voltage Source

- (a) The initial voltage across the capacitor shown in Figure 2 is $v_c(0+) = 0$. Find an expression for the voltage across the capacitor as a function of time, and sketch it to scale versus time.

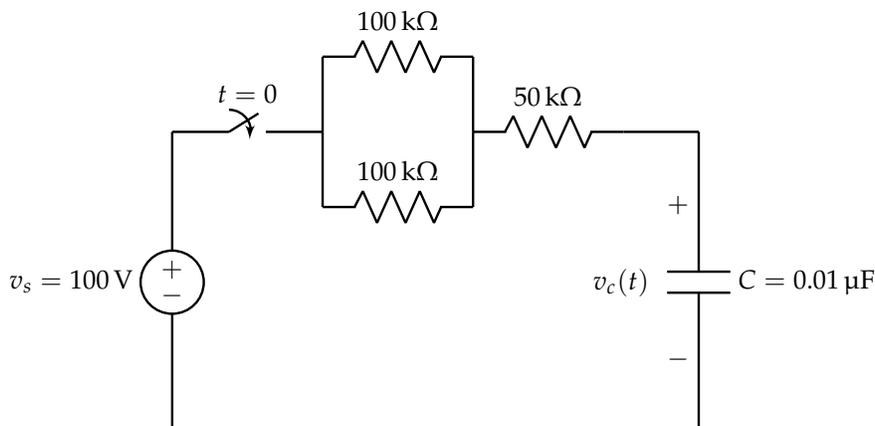
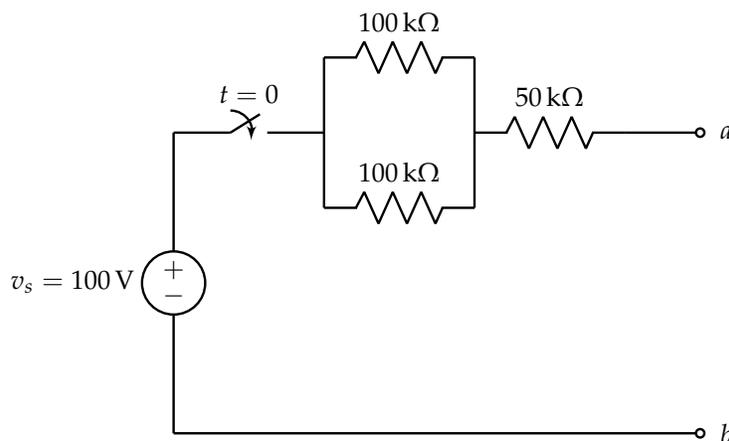


Figure 2: Circuit diagram.

(HINT: Consider simplifying the circuit using Thevenin equivalent circuits. That is, consider the following circuit, which is exactly Figure 2 without the capacitor:



Find a Thevenin equivalent circuit and use this to simplify Figure 2.)

- (b) Repeat part (a) for an initial voltage $v_c(0+) = -50$ V.

4. Two Capacitors

Consider the circuit shown in Figure 3. Prior to $t = 0$, $v_1 = 100$ V and $v_2 = 0$.

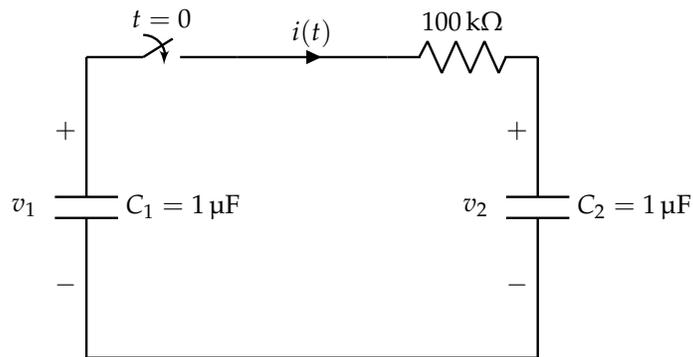


Figure 3: Circuit diagram.

- Immediately after the switch is closed, what is the value of the current (i.e., what is the value of $i(0+)$)?
- Write the KVL equation for the circuit in terms of the current and initial voltages. Take the derivative to obtain a differential equation.
- What is the value of the time constant in this circuit?
- Find an expression for the current as a function of time.
- Find the value that v_2 approaches as t becomes very large.

5. (Lab) Op-Amp Stability

In this question we will revisit the basic op-amp model that was introduced in EECS 16A and we will add a capacitance C_{out} to make the model more realistic (refer to figure 4). This question is also relevant to your lab projects. Now that we have the tools to do so, we will study the behavior of the op-amp in positive and negative feedback (refer to figure 5).

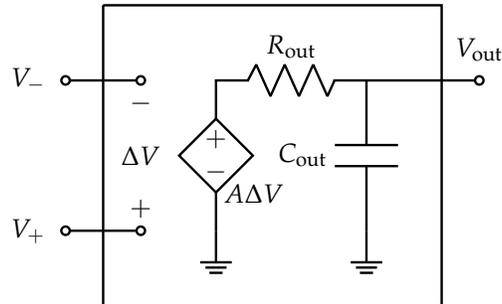
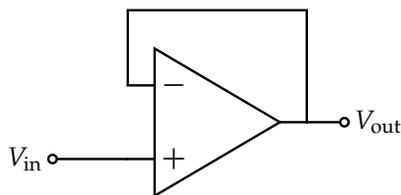
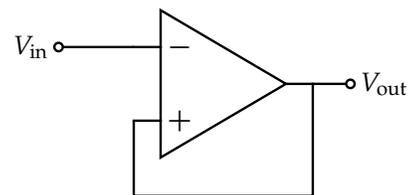


Figure 4: Op-amp model: $\Delta V = V_+ - V_-$



(a) Buffer in negative feedback



(b) "Buffer" in positive feedback that doesn't actually work as a buffer.

Figure 5: Op-amp in buffer configuration

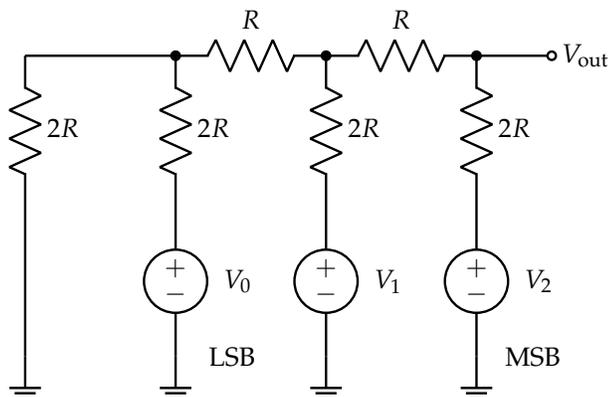
- Using the op-amp model in figure 4 and the buffer in negative-feedback configuration in figure 5a, **draw a combined circuit**. Remember that $\Delta V = V_+ - V_-$, the voltage difference between the positive and negative labeled input terminals of the op-amp. *NOTE:* Here, we have used the equivalent model for the op-amp gain. In more advanced analog circuits courses, it is traditional to use a controlled current source with a resistor in parallel instead.
- Let's look at the op-amp in negative feedback. From our discussions in EECS 16A, we know that the buffer in figure 5a should work with $V_{out} \approx V_{in}$ by the golden rules. **Write a differential equation for V_{out} by replacing the op-amp with the given model and show what the solution will be as a function of time for a static V_{in} . What does it converge to as $t \rightarrow \infty$?** Note: We assume the gain $A > 1$ for all parts of the question. For this part, you can assume the initial condition $V_{out}(0) = 0$.
- Next, let's look at the op-amp in positive feedback. We know that the configuration given in figure 5b is unstable and V_{out} will just rail. **Again, using the op-amp model in figure 4, show that V_{out} does not converge and hence the output will rail. For positive DC input $V_{in} > 0$, will V_{out} rail to the positive or negative side? Explain.** For this part, you can assume the initial condition $V_{out}(0) = 0$.

- (d) For an ideal op-amp, we can assume that it has an infinite gain, i.e., $A \rightarrow \infty$. **Under these assumptions, show that the op-amp in negative feedback behaves as an ideal buffer, i.e., $V_{\text{out}} = V_{\text{in}}$.**

6. (Lab) Digital-Analog Converter

A digital-analog converter (DAC) is one of the key interface components between the digital and the analog world, and also relevant for your lab projects. It is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R - $2R$ ladder. This DAC will help us generate the analog voltages from the digital representation, and later will also help us digitize the analog voltages when we will be building analog to digital interfaces in Lab 3, in part based on this ladder-DAC.

Here is the circuit for a 3-bit resistive DAC.



Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{DD}$, $V_1 = b_1 V_{DD}$, $V_2 = b_2 V_{DD}$, where V_{DD} is the supply voltage.

As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. b_0 is the least significant bit (LSB) and b_2 is the most significant bit (MSB). We will now analyze how this converter functions.

- Solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0 .
- If $b_2, b_1, b_0 = 0, 1, 1$, what is V_{out} ? Express your answer in terms of V_{DD} .
- If $b_2, b_1, b_0 = 1, 0, 1$, what is V_{out} ? Express your answer in terms of V_{DD} .
- If $b_2, b_1, b_0 = 1, 1, 0$, what is V_{out} ? Express your answer in terms of V_{DD} .
- If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out} ? Express your answer in terms of V_{DD} .
- Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out} .

Contributors:

- Anant Sahai.
- Nikhil Shinde.
- Druv Pai.
- Anish Muthali.
- Aditya Arun.
- Gaoyue Zhou.
- Ayan Biswas.

- Antroy Roy Chowdhury.
- Nikhil Jain.
- Chancharik Mitra.
- Edward Wang.
- Wahid Rahman.