Calculating the Singular Value Decomposition

Suppose we have a matrix A of dimension $m \times n$ ($n > m$) with rank r.

We can find the singular value decomposition (SVD)

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

with the following steps.

1. Find the eigenvalues λ_i of $A^T A$ and order them such that $\lambda_1 \geq \ldots \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

2. Find the orthonormal eigenvectors of $A^T A$, so that

$$A^T A \vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \ldots, r$$

   Note that the vectors must be orthonormal, that is $\vec{v}_i^T \vec{v}_i = 1$ and $\vec{v}_i^T v_j = 0$ for $i \neq j$.

3. Let $\sigma_i = \sqrt{\lambda_i}$ and set

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}, \quad i = 1, \ldots, r$$

Note: We will see later that real symmetric matrices $Q = Q^T$ have real eigenvalues and a set of real, orthonormal eigenvectors. Moreover if we can write $Q = R^T R$, the eigenvalues are non-negative.

1 SVD Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

a) Find the SVD of A.

b) Find the rank of A.

c) Find a basis for the nullspace of A.

d) Find a basis for the range (or columnspace) of A.

e) Create the SVD of A^T. What are the relationships between the answers to (a)-(d) for A and for A^T?