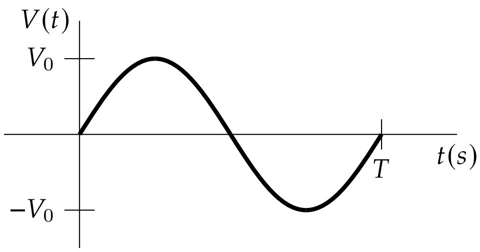
1 Phasors

We consider sinusoidal voltages and currents of a specific form:

Voltage	$v(t) = V_0 \cos(\omega t + \phi_v)$
Current	$i(t) = I_0 \cos(\omega t + \phi_i)$

where,

1. V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.
2. ω is the angular frequency of oscillation. ω is related to frequency by $\omega = 2\pi f$. Frequency f is the number of oscillation cycles that occur per second. If T is the *period* of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is $f = \frac{1}{T}$.



3. ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \text{Re}\left(e^{j\theta}\right) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = \text{Re}\left(V_0 e^{j(\omega t + \phi_v)}\right) = \text{Re}\left(V_0 e^{j(\omega t + \phi_v)}\right) = \text{Re}\left(V_0 e^{j\phi_v} e^{j\omega t}\right) \tag{1}$$

The part of this equation that does not depend on time is called the phasor:

$$V = V_0 e^{j\phi_v}$$

The complex conjugate of the phasor is given by:

$$\overline{V} = V_0 e^{-j\phi_v}$$

The phasor representation is a constant that contains the magnitude and phase information of the signal. The time-varying part of the signal does not need to be explicitly represented, because it is given by $e^{j\omega t}$, which is always implicit when using phasors. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like (16A style) circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: Phasor analysis only works for a steady-state solution; it does not work on analyses that require transients. We can also only use phasors if we know that all of our signals are sinusoids. We will see later in 16B (Module 4) why the latter is not too restrictive a condition.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where V is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \mathrm{Re}(Ve^{j\omega t})$$

The standard forms for voltage and current phasors are given below:

Voltage	$V = V_0 e^{j\phi_v}$
Current	$I = I_0 e^{j\phi_i}$

We define the *impedance* of a circuit component to be $Z = \frac{V}{I}$, where V and I represent the voltage across and the current through the component, respectively. For capacitors and inductors, this impedance will generally depend on $j\omega$.

Phasor Relationship for Resistors

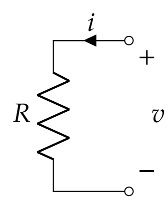


Figure 1: A simple resistor circuit

Consider a simple resistor circuit as in Figure 1, with current being

$$i(t) = I_0 \cos(\omega t + \phi) = \mathrm{Re}\left(I_0 e^{j\phi} e^{j\omega t}\right)$$

We see that $I = I_0 e^{j\phi}$. By Ohm's law,

$$\begin{aligned} v(t) &= i(t) \cdot R \\ &= I_0 R \cos(\omega t + \phi) \\ &= \mathrm{Re}\left(I_0 R e^{j\phi} e^{j\omega t}\right) \\ &= I_0 \cdot R \cdot \mathrm{Re}\left(e^{j\phi} e^{j\omega t}\right) \end{aligned}$$

Hence $V = I_0 R e^{j\phi}$ In the phasor domain,

$$V = IR$$

We usually refer to the impedance of the resistor, Z_R, in the phasor domain. Since $Z_R = R$, we can also write:

$$V = IZ_R$$

Phasor Relationship for Capacitors

Consider a capacitor circuit as in Figure 2, with voltage being

$$v(t) = V_0 \cos(\omega t + \phi)$$

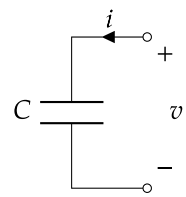


Figure 2: A simple capacitor circuit

By the capacitor equation, and expanding cos in the complex exponential domain:

$$\begin{aligned} v(t) &= V_0 \cos(\omega t + \phi) \\ &= \mathrm{Re}\left(V_0 e^{j\omega t + j\phi}\right) \\ &= \mathrm{Re}\left(V e^{j\omega t}\right) \end{aligned}$$

$$\begin{aligned} i(t) &= C \frac{d}{dt} v(t) \\ &= C \frac{d}{dt} \mathrm{Re}\left(V_0 e^{j\omega t} e^{j\phi}\right) \\ &= \mathrm{Re}\left(C V_0 \frac{d}{dt} e^{j\omega t} e^{j\phi}\right) \\ &= \mathrm{Re}\left(j\omega C V_0 e^{j\omega t} e^{j\phi}\right) \\ &= \mathrm{Re}\left(j\omega C V e^{j\omega t}\right) \end{aligned}$$

where

$$I = j\omega C V_0 e^{j\phi} = j\omega C V$$

Note that this works because we define $i(t) = \mathrm{Re}\left(I e^{j\omega t}\right)$ for our current phasor I.

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted Z_C, which is given by:

$$Z_C = \frac{V}{I} = \frac{1}{j\omega C}.$$

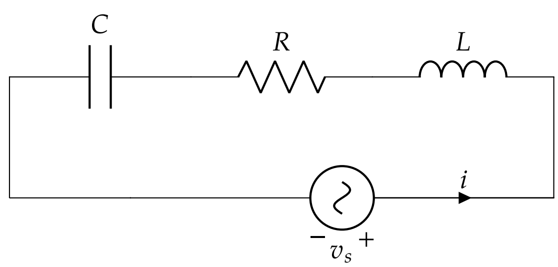
2 Inductor Impedance

Given the voltage-current relationship of an inductor $V = L\frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

3 RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = V_0 \cos\big(\omega t - \phi\big)$$



a) Write out the phasor representations of $v_s(t)$, and the impedances of R, C, and L.

b) Use Kirchhoff's laws to write down a loop equation relating the phasors in the previous part.

c) Solve the equation in the previous step for the current I. What is the magnitude and phase of the polar form of I?