Discussion 12A

The following notes are useful for this discussion: Note 18.

1. Jacobians and Linear Approximation

Recall that for a scalar-valued function $f(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ with vector-valued arguments, we can linearize the function at $(\vec{x}_\star, \vec{y}_\star)$:

$$\widehat{f}(\vec{x},\vec{y}) = f(\vec{x}_\star,\vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x}_\star,\vec{y}_\star)}{\partial x_i}(x_i - x_{i,\star}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x}_\star,\vec{y}_\star)}{\partial y_j}(y_j - y_{j,\star}). \tag{1}$$

In order to simplify this equation, we can define the following two vector quantities:

$$J_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \tag{2}$$

$$J_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix} \tag{3}$$

(a) When the function $\vec{f}(\vec{x},\vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a *vector* (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them as above:

$$\hat{\vec{f}}(\vec{x},\vec{y}) = \begin{bmatrix} \hat{f}_1(\vec{x},\vec{y}) \\ \hat{f}_2(\vec{x},\vec{y}) \\ \vdots \\ \hat{f}_m(\vec{x},\vec{y}) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_1 \cdot (\vec{x}-\vec{x}_\star) + J_{\vec{y}} f_1 \cdot (\vec{y}-\vec{y}_\star) \\ f_2(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_2 \cdot (\vec{x}-\vec{x}_\star) + J_{\vec{y}} f_2 \cdot (\vec{y}-\vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} f_m \cdot (\vec{x}-\vec{x}_\star) + J_{\vec{y}} f_m \cdot (\vec{y}-\vec{y}_\star) \end{bmatrix} \tag{4}$$

We can rewrite this in a clean way with the *Jacobian* of a vector-valued function:

$$J_{\vec{x}} \vec{f} = \begin{bmatrix} J_{\vec{x}} f_1 \\ J_{\vec{x}} f_2 \\ \vdots \\ J_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{5}$$

and similarly

$$J_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{6}$$

Then, the linearization becomes

$$\hat{\vec{f}}(\vec{x},\vec{y}) = \vec{f}(\vec{x}_\star,\vec{y}_\star) + J_{\vec{x}} \vec{f}(\vec{x}_\star,\vec{y}_\star) \cdot (\vec{x}-\vec{x}_\star) + J_{\vec{y}} \vec{f}(\vec{x}_\star,\vec{y}_\star) \cdot (\vec{y}-\vec{y}_\star). \tag{7}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $J_{\vec{x}} \vec{f}$, applying the definition above.

(b) **Evaluate the approximation of $\vec{f}$ using $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$, and compare with $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$. Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.**

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$. **Find $J_{\vec{x}}\vec{f}$ and $J_{\vec{y}}\vec{f}$.**

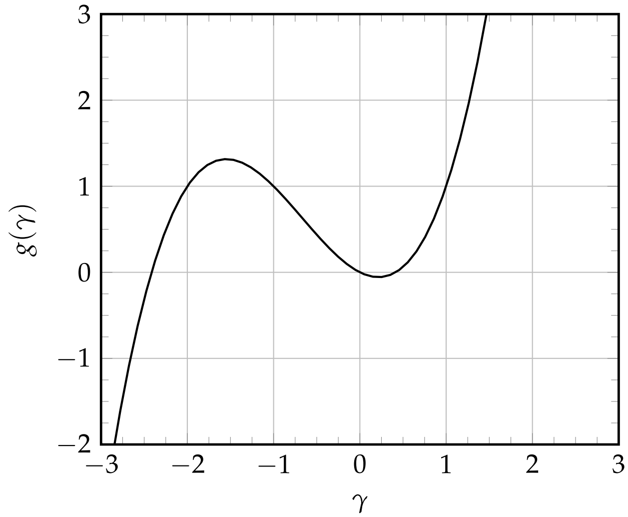
(d) **(PRACTICE)** Continuing the above part, **find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t))+u(t)\end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{8}$$

where $\vec{x}(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point $\vec{x}_\star$ is an operating point if $\vec{f}(\vec{x}_\star(t), u_\star(t)) = \vec{0}$.

(a) **If we have fixed $u_\star(t) = -1$, what values of γ and β will ensure $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{f}(\vec{x}(t), u(t)) = \vec{0}$?**

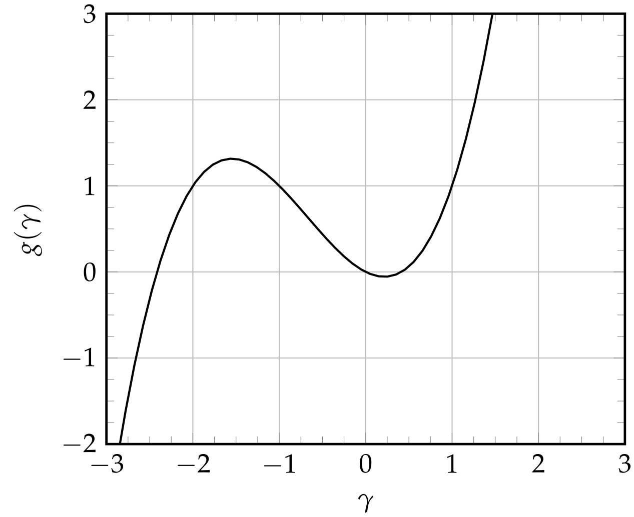
(b) Now that you have the three operating points, **linearize the system about the operating point $(\vec{x}_3^\star, u_\star)$ (that which has the largest value for γ).** Specifically, what we want is as follows. Let $\delta\vec{x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u_\star$. We can in principle write the linearized system for each operating point in the following form:

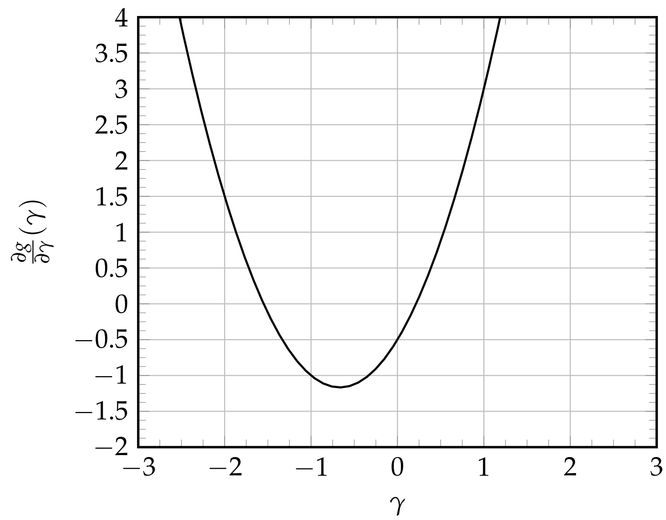
$$(\text{linearization about } (\vec{x}_i^\star, u_\star)) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}_i(t) = A_i\delta\vec{x}_i(t) + B_i\delta u(t) + \vec{w}_i(t) \tag{9}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_i **and** B_i.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





(c) **Which of the operating points are stable? Which are unstable?**

Contributors:

- Neelesh Ramachandran.
- Kuan-Yun Lee.
- Alex Devonport.
- Kumar Krishna Agrawal.