

The following notes are useful for this discussion: [Note 10](#), [Note 11](#), [Note 12](#)

1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

Solution: For notation's sake, let's write the system in the familiar form

$$\vec{x}[i + 1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i] \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

We have to calculate the eigenvalues of matrix A . Doing so, we find:

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = -2 \tag{4}$$

Since there exists a λ such that $|\lambda| \geq 1$ (in fact, both λ_1 and λ_2 satisfy this inequality), the system is unstable.

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (5), what is an equivalent representation for this system? Write your answer as $\vec{x}[i + 1] = A_{CL}\vec{x}[i]$ for some matrix A_{CL} .**

$$u[i] = [f_1 \quad f_2] \vec{x}[i] \tag{5}$$

HINT: If you're having trouble parsing the expression for $u[i]$, note that $[f_1 \quad f_2]$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] \tag{6}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} ([f_1 \quad f_2] \vec{x}[i]) \tag{7}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [f_1 \quad f_2] \right) \vec{x}[i] \tag{8}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i] \tag{9}$$

$$= \underbrace{\begin{bmatrix} f_1 & 1 + f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i]. \tag{10}$$

- (c) Find the appropriate state feedback constants, f_1, f_2 , that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Solution: From the previous part we have computed the closed loop system as

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i] \quad (11)$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) \quad (12)$$

$$= \det \left(\begin{bmatrix} f_1 - \lambda & 1 + f_2 \\ 2 & -1 - \lambda \end{bmatrix} \right) \quad (13)$$

$$= \lambda^2 + (1 - f_1)\lambda + (-f_1 - 2f_2 - 2) \quad (14)$$

We want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This means that we should choose the constants f_1 and f_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2} \right) \left(\lambda + \frac{1}{2} \right) = \lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4} \quad (15)$$

Thus, we can match the coefficients of λ in the polynomial above, which indicates we should choose f_1 and f_2 satisfying the following system of equations:

$$0 = 1 - f_1 \quad (16)$$

$$-\frac{1}{4} = -f_1 - 2f_2 - 2 \quad (17)$$

We can solve this two variable, two equation system and find that $f_1 = 1, f_2 = -\frac{11}{8}$.

Alternatively, we know what the eigenvalues are; we can plug in each λ into characteristic polynomial, and doing so will yield the same system of equations in f_1, f_2 .

- (d) Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?

Solution: Yes, the closed loop system has eigenvalues $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$, which means that A_{CL} satisfies our condition that all of its eigenvalues have magnitude less than 1.

- (e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (18). As before, we use $u[i] = [f_1 \ f_2] \vec{x}[i]$ to try and control the system.

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \quad (18)$$

Show that the resulting closed-loop state space matrix is

$$A_{CL} = \begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix} \quad (19)$$

Is it possible to stabilize this system?

Solution:

$$\vec{x}[i+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ f_2] \right) \vec{x}[i] \quad (20)$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \quad (21)$$

$$= \underbrace{\begin{bmatrix} f_1 & f_2 + 1 \\ f_1 + 2 & f_2 - 1 \end{bmatrix}}_{A_{CL}} \vec{x}[i] \quad (22)$$

Finding the eigenvalues λ :

$$0 = \det \left(\begin{bmatrix} f_1 - \lambda & f_2 + 1 \\ f_1 + 2 & f_2 - 1 - \lambda \end{bmatrix} \right) \quad (23)$$

$$= (f_1 - \lambda)(f_2 - 1 - \lambda) - (f_1 + 2)(f_2 + 1) \quad (24)$$

$$= f_1(f_2 - 1) - f_1\lambda - \lambda(f_2 - 1) + \lambda^2 - (f_1f_2 + f_1 + 2f_2 + 2) \quad (25)$$

$$= f_1f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1f_2 - f_1 - 2f_2 - 2 \quad (26)$$

$$= \lambda^2 + (1 - f_1 - f_2)\lambda - 2(1 + f_1 + f_2) \quad (27)$$

$$= (\lambda + 2)(\lambda - (1 + f_1 + f_2)) \quad (28)$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input. Since there will always be an eigenvalue with $|\lambda| \geq 1$, then we cannot stabilize the system.

- (f) **(PRACTICE)** Suppose you had a discrete, 2D, linear system with a real A matrix, and that you could modify both eigenvalues with feedback control (such as the system in eq. (1)). **Can you place the eigenvalues at complex conjugates, such that $\lambda_1 = a + jb, \lambda_2 = a - jb$ using only real feedback gains f_1, f_2 ? How about placing them at any arbitrary complex numbers, such that $\lambda_1 = a + jb, \lambda_2 = c + jd$?**

Solution: Recall that the eigenvalues will be the roots of the polynomial $a\lambda^2 + b\lambda + c$ for some real constants $a, b, c \in \mathbb{R}$. Since all of the coefficients are real, it is not possible to have arbitrary complex eigenvalues — they must either both be real or be complex conjugates. By the quadratic formula, the roots will be $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and the only possibly complex quantity can be $\sqrt{b^2 - 4ac}$. Suppose it is complex. Then, we can write $\sqrt{b^2 - 4ac} = j\gamma$ for $\gamma \in \mathbb{R}$. As such, the eigenvalues will be $\lambda_1 = \frac{-b + j\gamma}{2a}$ and $\lambda_2 = \frac{-b - j\gamma}{2a}$ which are complex conjugates.

2. Uncontrollability

Recall that, for a n -dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $\mathcal{C} = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ to be rank n .

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i + 1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \quad (29)$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (30)$$

- (a) **Is the system controllable?**

Solution:

$$\mathcal{C} = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad (31)$$

Since the controllability matrix \mathcal{C} only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space \mathbb{R}^3 .

(b) Show that we can write the i th state as

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (32)$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ ? If so, for what input sequence $u[i]$ up to $i = \ell - 1$?

Solution: We can write:

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i-1] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \quad (33)$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[i-1] \\ x_2[i-1] \\ x_3[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \quad (34)$$

$$= \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (35)$$

$$= \begin{bmatrix} 2^i x_1[0] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (36)$$

$$= \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \quad (37)$$

Note that in this expression for $\vec{x}[i]$, $x_1[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs u to get to $x_1[\ell] = -2$. Therefore,

we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any ℓ .

(c) (PRACTICE) Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some ℓ ? For what input sequence $u[i]$ for

$i = 0$ to $i = \ell - 1$?

HINT: Use the result for $\vec{x}[i]$ from the previous part.

Solution: We need $\ell = 1$ since $x_1[i] = 2^i x_1[0] = 2^i$. Hence,

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (38)$$

We realize that the first two entries of $\vec{x}[1]$ are exactly what we want. Thus, we have to choose $u[0]$ so the third entry is -2 . If we choose $u[0] = -1$, then we have reached our desired state.

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

(d) Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.

Solution:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \quad (39)$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] \quad (40)$$

$$= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} \quad (41)$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

after two timesteps. This means that we can reach any value for $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.

Alternative Solution:

Notice that we can write

$$\vec{x}[2] = A^2 \vec{x}[0] + ABu[0] + Bu[1] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + [B \quad AB] \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \quad (42)$$

Hence, $\vec{x}[2]$ will be $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ plus whatever is in the column space of

$$[B \quad AB] = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \quad (43)$$

This gives the same answer as before, i.e. that

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \vec{p} \quad (44)$$

where $\vec{p} \in \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

Any desired $x_2[i]$ and $x_3[i]$ that we can possibly reach can be obtained in only two or fewer timesteps. Hence, every reachable state can be written as

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ 0 \\ 0 \end{bmatrix} + \vec{p} \quad (45)$$

with \vec{p} defined as above. This will also tell us why the desired goal in part 2.b is unreachable.

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