Discussion 7A

The following notes are useful for this discussion: Note 10 and Note 11.

1. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Solution: The system is stable, as $\lambda = 0.9 \implies |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \tag{2}$$
$$x[1] = w[0] \tag{3}$$
$$x[2] = 0.9w[0] + w[1] \tag{4}$$
$$x[3] = 0.9^2 w[0] + 0.9w[1] + w[2] \tag{5}$$
$$\vdots \tag{6}$$
$$x[i] = \sum_{k=0}^{i-1} 0.9^k w[i-k-1]. \tag{7}$$

We can check that this form works by plugging it into our recursion:

$$x[i+1] = 0.9x[i] + w[i] = 0.9\left(\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right) + w[i] = \sum_{k=0}^{i-1} 0.9^{k+1} w[i-k-1] + w[i] = \sum_{k=0}^{i} 0.9^k w[i-k] \tag{8}$$

which is exactly what our formula predicts. So,

$$|x[i]| = \left|\sum_{k=0}^{i-1} 0.9^k w[i-k-1]\right| \leq \sum_{k=0}^{i-1} \left|0.9^k w[i-k-1]\right| = \sum_{k=0}^{i-1} 0.9^k \epsilon. \tag{9}$$

In the limit as $i \to \infty$, by the geometric series formula,

$$|x[i]| \leq \frac{\epsilon}{1-0.9} = 10\epsilon \tag{10}$$

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ, you want the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{11}$$

To do so, how would you pick f?
NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.
Solution: We can control the system to have any value of λ, as long as we're not limited on the values of f.

$$x[i+1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \tag{12}$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small λ by the choice of f.

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**
Solution: From eq. (11), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this λ, $f = -0.9$. In the limit as $i \to \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1-0} = \epsilon \tag{13}$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

(d) **What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control λ change?**
Solution: If our system were now,

$$x[i+1] = 3x[i] + u[i] + w[i], \tag{14}$$

the system would no longer be stable. However, we can still choose any λ using closed loop feedback. In this case, $f = \lambda - 3$.

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{15}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.
Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i] + \vec{w}[i]? \tag{16}$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL}? Will this work for any desired A_{CL}?
Solution: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix A_{CL}. As long as B is invertible (as given), we can define:

$$\begin{aligned}
\vec{x}[i+1] &= A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] && (17)\\
&= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] && (18)\\
&= (A + BF)\vec{x}[i] + \vec{w}[i] && (19)\\
&= A_{\text{CL}}\vec{x}[i] + \vec{w}[i] && (20)
\end{aligned}$$

Therefore, matching terms,

$$A + BF = A_{\text{CL}} \implies F = B^{-1}(A_{\text{CL}} - A). \tag{21}$$

2. Controlling states by designing sequences of inputs

Consider the following matrix, with a simple structure (what does it do when it acts on a vector?):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{23}$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $\ell \geq 0$ (Note that ℓ denotes total sequence length.) The key is that we want to be in this specific state at this specific timestep, ℓ. **What is the smallest ℓ such that this is possible? What is our choice of sequence of inputs $u[i]$?**

Solution: To ease notation, let

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \\ x_3[i] \\ x_4[i] \end{bmatrix}. \tag{24}$$

Writing out expressions for $x[i]$ we get:

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix}, \tag{25}$$

$$\vec{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix}, \qquad \vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}, \tag{26}$$

and if $i \geq 4$,

$$\vec{x}[i] = \begin{bmatrix} u[i-4] \\ u[i-3] \\ u[i-2] \\ u[i-1] \end{bmatrix}. \tag{27}$$

Hence, the smallest ℓ is equal to 4 (we applied 4 inputs), with $u[0] = [1]$, $u[1] = [2]$, $u[2] = [3]$, $u[3] = [4]$.

(b) **What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest ℓ and what is our choice of $u[i]$?**

Solution: Looking over our expressions for $x[i]$ from the previous part, we see that the earliest ℓ whose expression can be set to the desired state is $\ell = 1$ requiring $u[0] = 4$.

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}. \tag{28}$$

(c) **If we start from** $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$**, what is smallest** ℓ **such that** $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$**, what is corresponding** $u[i]$**?**

Solution: Looking over our expressions for x[i], we see that the earliest ℓ whose expression can be set to the desired state in this case is $\ell = 4$ requiring $u[0] = 1$, $u[1] = 2$, $u[2] = 3$, $u[3] = 4$.

(d) **If you would like to make sure that at time** ℓ **we are at** $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ **for the state, what controls could you use to get there? How big does** ℓ **have to be for this strategy to work?**

Solution: As you might notice, using inputs $u[0] = a, u[1] = b, u[2] = c, u[3] = d$, we can get to any desired state $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence with $\ell = 4$, we can guarantee that the $\vec{x}[\ell]$ is our desired state.

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