Discussion 6B

The following notes are useful for this discussion: Note 9, Note 10

1. System Identification by Means of Least Squares

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at timestep i is $x[i]$, the input applied at timestep i is $u[i]$ and $w[i]$ represents some (small) external disturbance that also participated at timestep i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to ℓ and also measurements for the controls $u[i]$ from $i = 0$ to $\ell - 1$. Further assume $\ell \geq 2$.

Show that we can set up a linear system as in eq. (2) to find constants a and b. How do we solve this system?

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{2}$$

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \tag{3}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

(d) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{4}$$

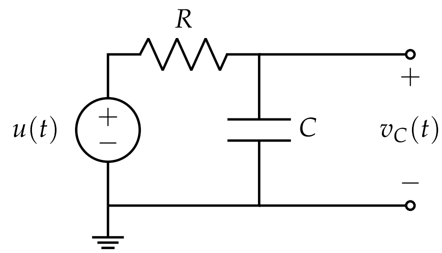
How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$**?** Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). *Hint: What work/computation can we reuse across the two problems?*

2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).

This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{5}$$



Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a+b| \leq |a| + |b|$, and the triangle inequality for integrals, i.e. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

(b) **(PRACTICE)** Now, suppose that in the circuit of part 2.a we replaced the resistor with an inductor as in fig. 1.

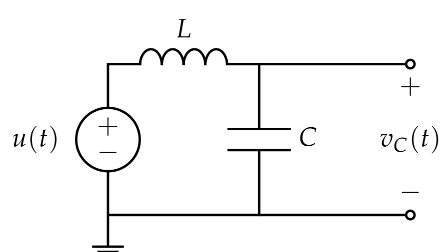


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1\,\mathrm{mH}$. **Repeat part 2.a for the new circuit (with an inductor). Consider the following process to arrive at the result:**

i. **Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is** $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ **with the initial condition being** $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$**.**

ii. **Solve the matrix differential equation, using diagonalization if needed. Show that the**

diagonalized system has a solution

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC} e^{j\frac{1}{\sqrt{LC}}t} \int_0^t e^{-j\frac{1}{\sqrt{LC}}\theta} u(\theta)\, d\theta \\ \frac{1}{2LC} e^{-j\frac{1}{\sqrt{LC}}t} \int_0^t e^{j\frac{1}{\sqrt{LC}}\theta} u(\theta)\, d\theta \end{bmatrix} \tag{6}$$

where $\vec{y}(t) = V^{-1}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ **for change of basis matrix** V**.** You may use the fact that the eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(j\frac{1}{\sqrt{LC}}, \begin{bmatrix} -j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$ and $\left(-j\frac{1}{\sqrt{LC}}, \begin{bmatrix} j\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \right)$.

iii. **Apply a similar process from part 2.a to show that, if we have a bounded input** $u(t)$**, then the system can grow unboundedly.** When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) = 2\cos\left(\frac{1}{\sqrt{LC}}\right) = e^{j\frac{1}{\sqrt{LC}}t} + e^{-j\frac{1}{\sqrt{LC}}t}$ [1]. *HINT: You may use the fact that* $i_L(t) = y_1(t) + y_2(t)$.

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ *and* $i_L(t)$ *as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.*

(c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{7}$$

[1] The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up".

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