EECS 16B Designing Information Systems and Devices II UC Berkeley Spring 2022

Discussion 4B

The following notes are useful for this discussion: Note 6.

1. Transfer Function Practice

Transfer functions take an input phasor and "transform" it into an output phasor. Most of the work we will do with transfer functions is analyzing how it will "respond" to a specific kind of input. We will also design our own transfer functions using common circuit components such as resistors, inductors, and capacitors to achieve some specified behavior. A block diagram of a transfer function is represented in fig. 1. In this discussion, we will learn how to derive $H(j\omega)$ from a given circuit, and we will analyze how it behaves for certain values of ω.

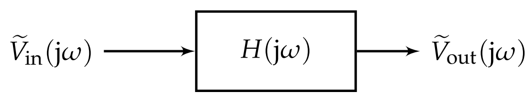
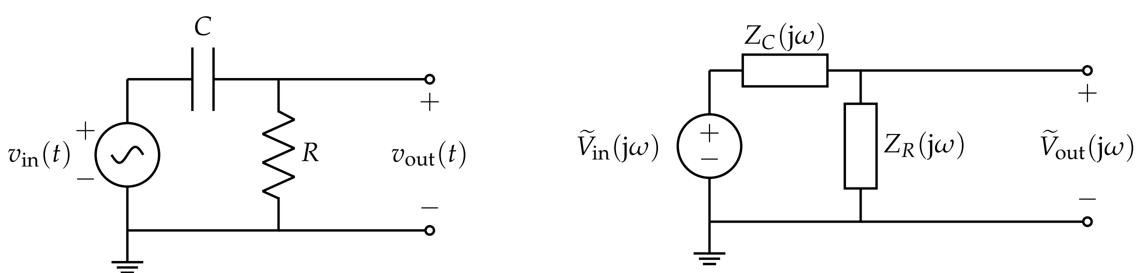


Figure 1: Transfer Function Block Diagram

Recall that $Z_L = j\omega L$ and $Z_C = \frac{1}{j\omega C}$. For large ω, $|Z_L| = \omega L$ becomes large (and becomes small for small ω). On the other hand, for large ω, $|Z_C| = \frac{1}{\omega C}$ becomes small (and becomes large for small ω).

In this problem, you'll be deriving some transfer functions on your own. **For each circuit, determine the transfer function $H(j\omega) = \frac{\widetilde{V}_{\text{out}}(j\omega)}{\widetilde{V}_{\text{in}}(j\omega)}$. How does $|H(j\omega)|$ respond as $\omega \to 0$ (low frequencies), as $\omega \to \infty$ (high frequencies)? Is the circuit a high-pass filter, low-pass filter, or band-pass filter?** As practice, sketch a graph (on a log-log plot) of $|H(j\omega)|$.

(a) **RC circuit:**



(a) Circuit in "time domain"

(b) Circuit in "phasor domain"

Solution: We'll use the voltage divider formula to find $\widetilde{V}_{\text{out}}(j\omega)$:

$$\widetilde{V}_{\text{out}}(j\omega) = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\text{in}}(j\omega) \tag{1}$$

Recalling the expression for the impendances, we note that for the resistor $Z_R = R$, and for the capacitor $Z_C = \frac{1}{j\omega C}$. Plugging in the impedances gives

$$H(j\omega) = \frac{\widetilde{V}_{\text{out}}(j\omega)}{\widetilde{V}_{\text{in}}(j\omega)} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega}{j\omega + \frac{1}{RC}} \tag{2}$$

At low frequencies, i.e with $\omega \ll \frac{1}{RC}$ we have

$$\lim_{\omega \to 0} |H(j\omega)| = \lim_{\omega \to 0} \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}} = 0 \tag{3}$$

At high frequencies with $\omega \gg \frac{1}{RC}$ we have

$$\begin{aligned} \lim_{\omega \to \infty} |H(j\omega)| &= \lim_{\omega \to \infty} \frac{\omega}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}} &(4)\\ &= \lim_{\omega \to \infty} \frac{1}{\sqrt{1 + \left(\frac{1}{\omega RC}\right)^2}} &(5)\\ &= 1 &(6) \end{aligned}$$

So this circuit is a *high-pass filter.* The magnitude and phase plots are depicted in fig. 5.

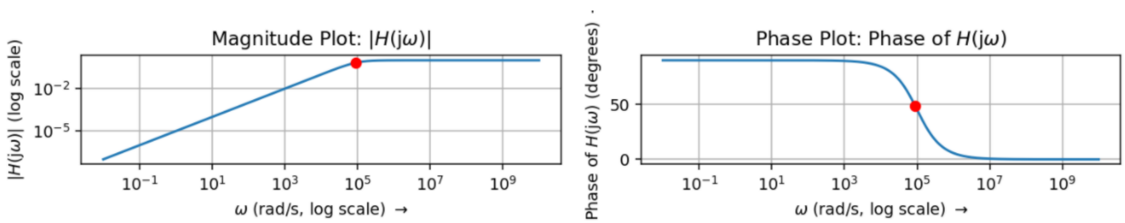
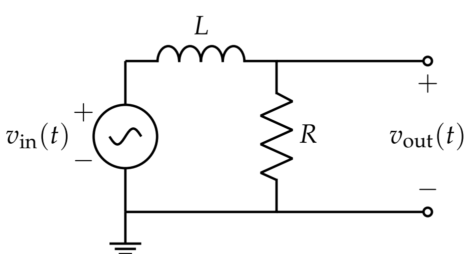
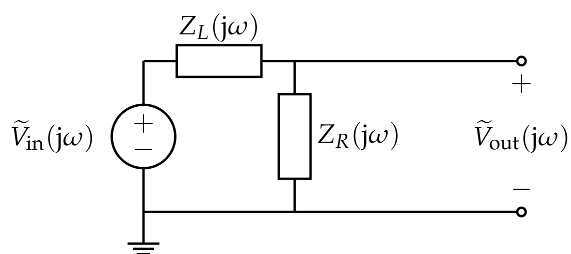


Figure 3: Magnitude and Phase Plot for RC Circuit

(b) **LR circuit:**



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: The strategy is the same as the previous part, using the voltage divider formula, i.e. ,

$$\widetilde{V}_{out}(j\omega) = \frac{Z_R}{Z_R + Z_L} \widetilde{V}_{in}(j\omega)$$

A similar manipulation to the previous part gives

$$H(j\omega) = \frac{\widetilde{V}_{out}(j\omega)}{\widetilde{V}_{in}(j\omega)} = \frac{R}{R + j\omega L} = \frac{\frac{R}{L}}{\frac{R}{L} + j\omega} \tag{7}$$

At low frequencies, i.e with $\omega \ll \frac{R}{L}$ we have

$$\lim_{\omega \to 0} |H(j\omega)| = \lim_{\omega \to 0} \frac{\frac{R}{L}}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}} = 1 \tag{8}$$

while at high frequencies with $\omega \gg \frac{R}{L}$, we have

$$\lim_{\omega\to\infty} |H(j\omega)| = \lim_{\omega\to\infty} \frac{\frac{R}{L}}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}} = 0 \tag{9}$$

So this circuit is a *low-pass filter*. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor. The effect of this change was to reverse the low-frequency and high-frequency behavior of the circuit! Another example of the complementarity of capacitors and inductors. The magnitude and phase plots are depicted in fig. 5.

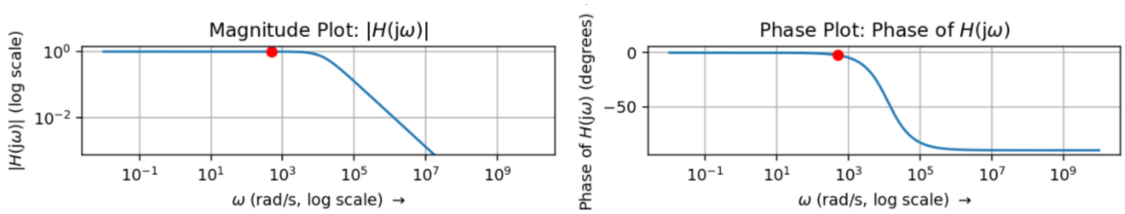
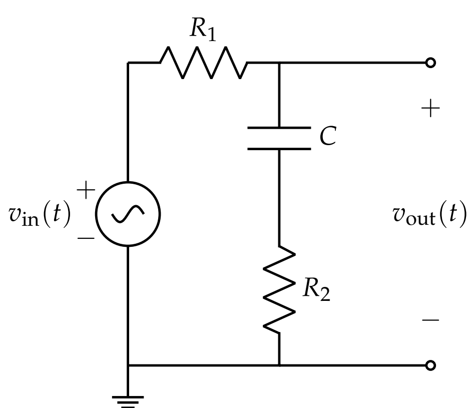
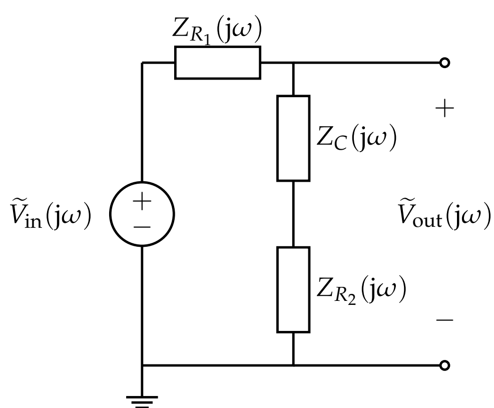


Figure 5: Magnitude and Phase Plot for LR Circuit

(c) **(PRACTICE) RCR circuit:**



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: Even though there are three components instead of two, we can still use the voltage divider formula by treating R_2 and C as a single impedance given by $Z = Z_C + Z_{R_2}$, giving us $Z = R_2 + \frac{1}{j\omega C}$. This would give us

$$\widetilde{V}_{out}(j\omega) = \frac{Z}{Z_{R_1} + Z}\widetilde{V}_{in}(j\omega) \tag{10}$$

Then, the transfer function is

$$H(j\omega) = \frac{\widetilde{V}_{out}(j\omega)}{\widetilde{V}_{in}(j\omega)} = \frac{R_2 + \frac{1}{j\omega C}}{R_1 + R_2 + \frac{1}{j\omega C}} = \frac{1 + j\omega R_2 C}{1 + j\omega C(R_1 + R_2)} \tag{11}$$

At low frequencies, we have

$$\lim_{\omega\to 0} |H(j\omega)| = \lim_{\omega\to 0} \frac{\sqrt{1 + (\omega R_2 C)^2}}{\sqrt{1 + (\omega C(R_1 + R_2))^2}} = 1 \tag{12}$$

while at high frequencies, we have

$$\lim_{\omega\to\infty} |H(j\omega)| = \lim_{\omega\to\infty} \frac{\sqrt{1+(\omega R_2 C)^2}}{\sqrt{1+(\omega C(R_1+R_2))^2}} \tag{13}$$

$$= \lim_{\omega\to\infty} \frac{\sqrt{\frac{1}{\omega^2}+(R_2 C)^2}}{\sqrt{\frac{1}{\omega^2}+(C(R_1+R_2))^2}} \tag{14}$$

$$= \frac{CR_2}{C(R_1+R_2)} = \frac{R_2}{R_1+R_2} \tag{15}$$

So at high frequencies, this circuit behaves like a regular voltage divider with just R_1 and R_2, as if the capacitor had vanished. This circuit is like a combination of a low-pass filter and a voltage divider: low frequency inputs are preserved, and high-frequency signals are diminished. The magnitude and phase plots are depicted in fig. 7.

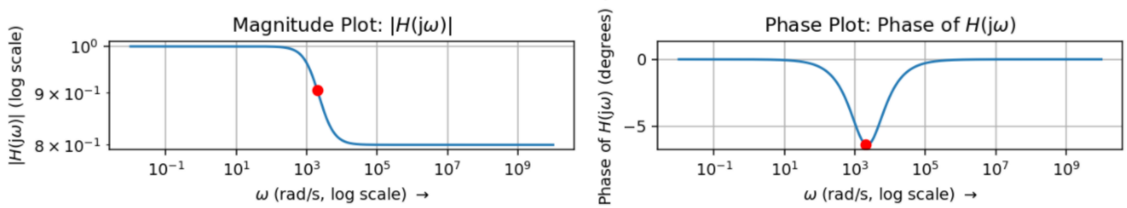


Figure 7: Magnitude and Phase Plot for RCR Circuit

(d) **Assuming $v_{in}(t) = 12\sin(\omega_{in}t)$ compute the $v_{out}(t)$ using the transfer function computed in part 1.a.** For this part, we assume that $R = 1\,\mathrm{k}\Omega, L = 25\,\mu\mathrm{H}, C = 10\,\mu\mathrm{F}, \omega_{in} = 100\,\mathrm{rad/s}$. In words, what is the effect of the transfer function in part 1.a on the magnitude of the input signal? Determine whether this behavior is expected given the input signal.

Solution: To get $v_{out}(t)$, we must first convert $v_{in}(t)$ into phasor domain to get $\widetilde{V}_{in}(j\omega)$, then apply the transfer function to get $\widetilde{V}_{out}(j\omega)$, and then convert back to time domain to get $v_{out}(t)$. To convert from time domain to phasor domain, recall the following procedure which uses Euler's method:

$$v_{in}(t) = v_0\cos(\omega t+\phi) \tag{16}$$

$$= \frac{v_0}{2}\left(e^{j\omega t+j\phi} + e^{-j\omega t-j\phi}\right) \tag{17}$$

$$= \frac{v_0 e^{j\phi}}{2}e^{j\omega t} + \frac{v_0 e^{-j\phi}}{2}e^{-j\omega t} \tag{18}$$

$$\implies \widetilde{V}_{in}(j\omega) = \frac{v_0 e^{j\phi}}{2} \tag{19}$$

Firstly, note that $\sin(x) = \cos\left(x-\frac{\pi}{2}\right)$, so we can write $v_{in} = 12\sin(\omega t)$ as $v_{in} = 12\cos\left(\omega t - \frac{\pi}{2}\right)$. Pattern matching with the result from eq. (19) (with $v_0 = 12$ and $\phi = -\frac{\pi}{2}$),

$$\widetilde{V}_{in}(j\omega) = 6e^{-j\frac{\pi}{2}} \tag{20}$$

Now, we can find $\widetilde{V}_{out}(j\omega)$ by multiplying the transfer function with the output phasor. Note that we have to evaluate the transfer function at $\omega = \omega_{in} = 100\,\mathrm{rad/s}$ since that is the input angular frequency:

$$H(j\omega_{in}) = \frac{j100}{\frac{1}{10^3*10^{-5}} + j100} \tag{21}$$

$$= \frac{\mathrm{j}}{1+\mathrm{j}} \tag{22}$$

We will write $H(\mathrm{j}\omega_{\mathrm{in}})$ in the form $|H(\mathrm{j}\omega_{\mathrm{in}})|\mathrm{e}^{\mathrm{j}\angle H(\mathrm{j}\omega_{\mathrm{in}})}$, so that multiplying with $\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)$ will be easier. First, to find $|H(\mathrm{j}\omega_{\mathrm{in}})|$:

$$|H(\mathrm{j}\omega_{\mathrm{in}})| = \left|\frac{\mathrm{j}}{1+\mathrm{j}}\right| = \frac{1}{\sqrt{2}} \tag{23}$$

Next, to find $\angle H(\mathrm{j}\omega_{\mathrm{in}})$:

$$\angle H(\mathrm{j}\omega_{\mathrm{in}}) = \angle(\mathrm{j}) - \angle(1+\mathrm{j}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \tag{24}$$

Hence, $H(\mathrm{j}\omega_{\mathrm{in}}) = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}$, and

$$\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega_{\mathrm{in}}) = H(\mathrm{j}\omega_{\mathrm{in}})\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega_{\mathrm{in}}) = 3\sqrt{2}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} \tag{25}$$

The last step is changing back to the time domain. Recall, that

$$v_{\mathrm{out}}(t) = \widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)\mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)}\mathrm{e}^{-\mathrm{j}\omega t} \tag{26}$$

$$= 2|\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)|\cos\left(\omega t + \angle\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)\right) \tag{27}$$

Substituting the values from eq. (25) we recover

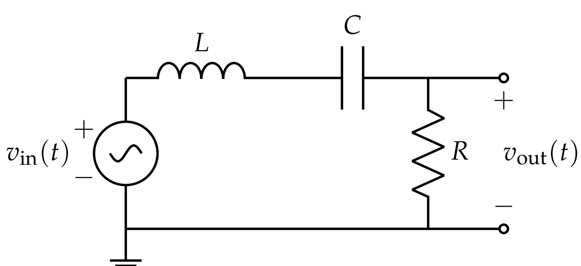
$$v_{\mathrm{out}}(t) = 6\sqrt{2}\cos\left(\omega_{\mathrm{in}}t - \frac{\pi}{4}\right) \tag{28}$$

The circuit from part **1.**a was a high pass filter, so we expect input signals with ω_{in} values closer to 0 to have lower magnitude once they are "passed through" the transfer function.

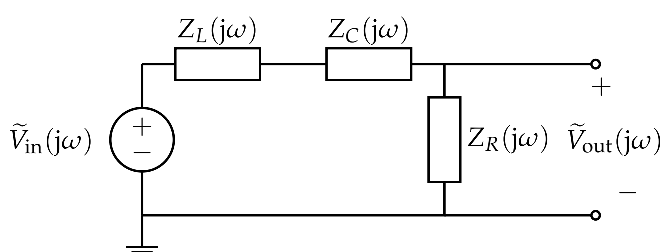
(e) **Visualizing Transfer functions:**
In this part, we visualize the transfer function for different types of circuits in a Jupyter Notebook.

2. Band-Pass Filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

(a) **Write down the transfer function $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)}$ for this circuit.**

Solution: Using the same voltage divider rule we've used in the past, $\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)$ is:

$$\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega) = \widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)\frac{Z_R}{Z_R + Z_L + Z_C} \tag{29}$$

$$= \widetilde{V}_{\text{in}} \frac{R}{R + \text{j}\left(\omega L - \frac{1}{\omega C}\right)} \tag{30}$$

$$\implies H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} \tag{31}$$

$$= \frac{R}{R + \text{j}(\omega L - \frac{1}{\omega C})} \tag{32}$$

(b) Consider the inductor, capacitor, and resistor connected in series. **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\text{j}\omega) = A(\omega) + \text{j}X(\omega)$, where $A(\omega)$ and $X(\omega)$ are real valued functions of ω. At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *natural frequency*.)

Solution: Recall that the series impedance is the denominator of the voltage divider formula. From the previous part, $Z_{RLC} = Z_R + Z_L + Z_C = R + \text{j}(\omega L - \frac{1}{\omega C})$. Thus, $A(\omega) = R$ and $X(\omega) = \omega L - \frac{1}{\omega C}$.

Now, we can proceed to find ω_n.

$$X(\omega_n) = \omega_n L - \frac{1}{\omega_n C} = 0 \tag{33}$$

Multiplying both sides by ω_n:

$$\omega_n^2 L - \frac{1}{C} = 0 \tag{34}$$

$$\omega_n = \frac{1}{\sqrt{LC}}. \tag{35}$$

(c) **Find an expression for $|H(\text{j}\omega)|$. What is $|H(\text{j}\omega_n)|$? What is $|H(\text{j}\omega_n/10)|$? What is $|H(\text{j}10\omega_n)|$?** Rank the three quantities: $|H(\text{j}\omega_n)|$, $|H(\text{j}\omega_n/10)|$, $|H(\text{j}10\omega_n)|$. What do you think the magnitude plot looks like?

Solution: We can compute $|H(\text{j}\omega)|$ as follows:

$$|H(\text{j}\omega)| = \left| \frac{R}{R + \text{j}(\omega L - \frac{1}{\omega C})} \right| \tag{36}$$

$$= \frac{|R|}{\left|R + \text{j}(\omega L - \frac{1}{\omega C})\right|} \tag{37}$$

$$= \frac{R}{\sqrt{\left(R + \text{j}(\omega L - \frac{1}{\omega C})\right)\left(R - \text{j}(\omega L - \frac{1}{\omega C})\right)}} \tag{38}$$

$$= \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \tag{39}$$

Note that the only part that depends on ω is the $X(\omega) = \omega L - \frac{1}{\omega C}$ term in the denominator. At $\omega = \omega_n$, this term is 0. Hence,

$$|H(\text{j}\omega_n)| = \frac{R}{\sqrt{R^2}} = 1 \tag{40}$$

At $\omega = \omega_n/10$, $X(\omega) = -9.9\sqrt{\frac{L}{C}}$. Similarly, at $\omega = 10\omega_n$, we have $X(\omega) = 9.9\sqrt{\frac{L}{C}}$. This means that $X(\omega_n/10)^2 = X(10\omega_n)^2$. Therefore,

$$|H(j\omega_n/10)| = |H(j10\omega_n)| = \frac{R}{\sqrt{R^2 + 9.9^2\frac{L}{C}}} < 1 \tag{41}$$

Thus, we would expect the graph of $|H(j\omega)|$ to sharply peak at $\omega = \omega_n$ and decrease for $\omega > \omega_n$ and $\omega < \omega_n$. Specifically, $|H(j\omega_n)| > |H(j\omega_n/10)| = |H(j10\omega_n)|$. This transfer function only "lets through" input signals with angular frequency $\omega = \omega_n$ and attenuates everything else.

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