1 System Identification and Linear Control

A scalar discrete-time system has the following dynamics:

$$x(t+1) = \lambda x(t) + g(u(t)), \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ not necessarily linear.

a) If g is approximated to order 2 around the operating point $u^* = 0$, so that

$$x(t+1) \approx \lambda x(t) + \beta_0 + \beta_1 u(t) + \beta_2 u^2(t), \tag{2}$$

what should β_0, β_1, and β_2 be?

Answer

From the MacLaurin series,

$$g(u) = g(0) + g'(0)u + \frac{1}{2}g''(0)u^2 + \left(\text{higher-order terms}\right). \tag{3}$$

We assume the argument of g to be small enough that its higher powers vanish.

b) Suppose that $x(0) = 0$. We apply a sequence of inputs

$$\vec{u} = \left(u(0), u(1), \ldots, u(N-1)\right) \tag{4}$$

and observe states $x(1), x(2), \ldots, x(N)$. Derive the least-squares estimates of λ, β_0, β_1, and β_2.

Answer

Notice that we may write $N-1$ system recurrences simultaneously as follows:

$$\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \lambda + \begin{bmatrix} 1 & u(0) & u^2(0) \\ 1 & u(1) & u^2(1) \\ \vdots & \vdots & \vdots \\ 1 & u(N-1) & u^2(N-1) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} x(0) & 1 & u(0) & u^2(0) \\ x(1) & 1 & u(1) & u^2(1) \\ \vdots & \vdots & \vdots & \vdots \\ x(N-1) & 1 & u(N-1) & u^2(N-1) \end{bmatrix} \begin{bmatrix} \lambda \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} \tag{6}$$

Give these vectors and matrices new names.

$$y = Dp \tag{7}$$

$$\hat{p} = \left(D^\top D\right)^{-1} D^\top y \tag{8}$$

2 Scalar feedback control

Suppose that x has the following discrete-time dynamics:

$$x(t+1) = \lambda x(t) + bu(t), \quad x(0) = x_0 \tag{9}$$

a) Assuming that $x_0 = 1$ and $u = 0$, sketch $x(t)$ for a few time steps for $\lambda \in \{-2, -1, 0, 1, 2\}$.

b) What qualifications for λ will result in convergence of x? A scalar system having such a λ is called *stable*.

Answer

$|\lambda|$ strictly less than 1.

c) If these system were the discretization from a continuous system at interval T, then $\lambda = e^{\lambda_{\text{cont}}T}$ and $b = b_{\text{cont}}\frac{e^{\lambda_{\text{cont}}T}-1}{\lambda_{\text{cont}}}$. Use these facts to relate convergence properties of continuous- and discrete-time linear systems. (A stable continuous-time scalar system has an eigenvalue with a strictly negative real part.)

Answer

This can be made intuitive if λ_{cont} is parameterized in Cartesian form as $\alpha + j\omega$, so that the exponential $e^{\lambda_{\text{cont}}t}$ contains an oscillation at frequency ω, scaled within the envelope $e^{\alpha t}$.

If $\alpha < 0$, then

$$\left|e^{\lambda_{\text{cont}}T}\right| = \left|e^{\alpha + j\omega}\right|^T \tag{10}$$

$$= \left|e^{\alpha}\right|^T \left|e^{j\omega}\right|^T \tag{11}$$

$$= \left|e^{\alpha}\right|^T \tag{12}$$

$$< 1^T = 1 \tag{13}$$

d) If $u(t) = u_0$ and x is stable, what does x converge to? Sketch stable trajectories of x for $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Answer

Solve for states that statisfy $x(t+1) = x(t)$.

$$x = \lambda x + bu_0 \tag{14}$$

$$x = \frac{bu_0}{1-\lambda} \tag{15}$$

Notice that this equilibrium is approximately bu_0 if $\lambda \approx 0$, and that it grows without bound as $|\lambda| \to 1$.

e) If $x(t+1) = \lambda x(t) + bu(t)$ is unstable, describe feedback laws $u(t) = kx(t)$ that stabilize the equilibrium $x = 0$.

Answer

Substituting $u(t) = kx(t)$,

$$\begin{aligned} x(t+1) &= \lambda x(t) + bkx(t) && (16) \\ &= (\lambda + bk)\, x(t) && (17) \end{aligned}$$

For stability of $x = 0$ we require

$$\begin{aligned} |\lambda + bk| &< 1 && (18) \\ -1 < \lambda + bk &< 1 && (19) \\ -1 - \lambda < bk &< 1 - \lambda && (20) \end{aligned}$$

Thus the stability criterion on k is

$$\begin{cases} -\dfrac{1+\lambda}{b} < k < \dfrac{1-\lambda}{b}, & b > 0 \\ \dfrac{1-\lambda}{b} < k < -\dfrac{1+\lambda}{b}, & b < 0 \end{cases} \tag{21}$$