1 Conditions for Equilibria

Continuous-Time Systems

Let us take a closer look at the conditions for a linear system represented by the differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{1}$$

From the get-go we see that $(\vec{x}^*, \vec{u}^*) = (\vec{0}, \vec{0})$ must be an equilibrium point. This is since the system is at rest. Now if we put in a constant input $\vec{u}^*$ then to solve for equilibria, we get the following system of equations

$$A\vec{x} + B\vec{u}^* = \vec{0} \tag{2}$$

To solve for the states $\vec{x}$ in which the system would be in equilibrium, our analysis boils down to whether the square matrix A is invertible [1].

a) If A is invertible, then there is a unique equilibrium point $\vec{x}^* = -A^{-1}B\vec{u}^*$.

b) If A is non-invertible, depending on the range of A, we have two scenarios.

- If $B\vec{u} \in \text{Col}(A)$ then we will have infinitely many equilibrium points.
- If $B\vec{u} \notin \text{Col}(A)$ then the system has no solution and we will have no equilibrium points.

Discrete-Time Systems

Now let's take a look at the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t) \tag{3}$$

Again we see that $(\vec{0}, \vec{0})$ is an equilibrium point but notice that the conditions for equilibria are different for discrete-time systems. A system is in equilibrium if it is not changing. In otherwords, this means that $\vec{x}^*(t+1]) = \vec{x}^*(t)$ therefore, for a constant input $\vec{u}^*$ we get the following system of equations

$$\vec{x} = A\vec{x} + B\vec{u}^* \implies (I - A)\vec{x} = B\vec{u}^* \tag{4}$$

The conditions for equilibria now depend on the matrix $I - A$ being invertible instead of the matrix A.

[1] This should be review from 16A/54, but we restate it here since it isn't quite obvious when A is singular or non-invertible. Normally a singular matrix has infinite solutions but take the system $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This leads to a contradiction that $x_1 = 0 \neq 1$.

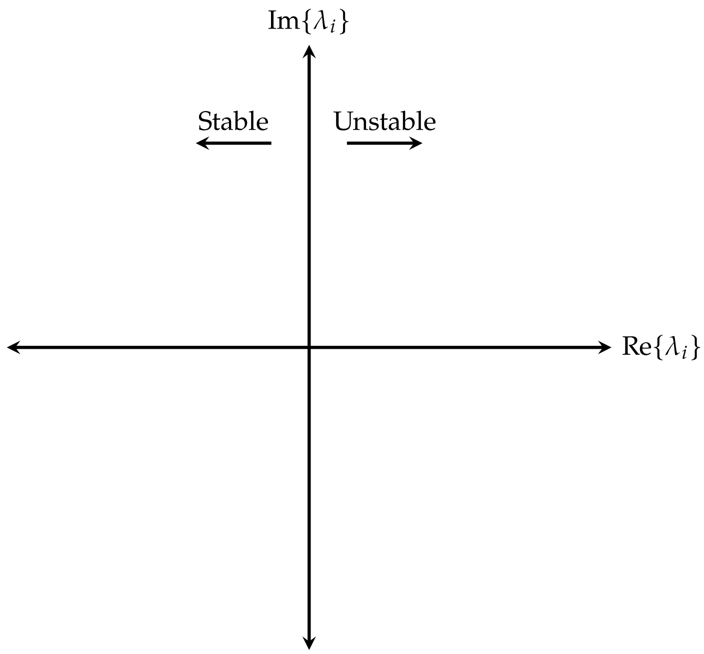
2 Stability

Continuous time systems

A continuous time system is of the form:

$$\frac{d\vec{x}}{dt}(t) = A\vec{x}(t) + B\vec{u}(t)$$

This system is stable if $\mathrm{Re}\{\lambda_i\} < 0$ for all λ_i , where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie to the left of $\mathrm{Re}\{\lambda_i\} = 0$, then the system is stable.



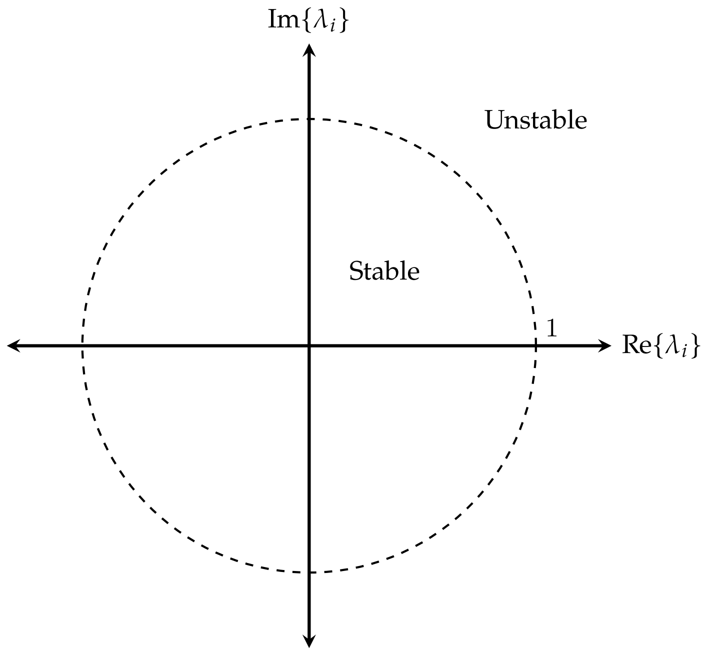
If $\mathrm{Re}\{\lambda_i\} \geq 0$, the system is unstable in the context of BIBO stability.

Discrete time systems

A discrete time system is of the form:

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

This system is stable if $|\lambda_i| < 1$ for all λ_i, where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie within (not on) the unit circle, then the system is stable.



If $|\lambda| \geq 1$, we say the system is unstable in the context of Bounded-Input Bounded-Output (BIBO) stability.

3 Jacobian Warm-Up

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - e^{x_2^2} \\ x_1^2 + \sin(x_1)x_2^2 \\ \log(1 + x_1^2) \end{bmatrix}$$

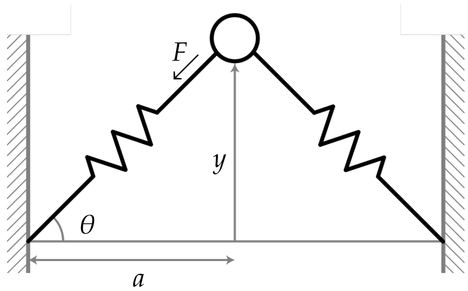
Calculate its Jacobian.

Answer

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\vec{x}} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & -2x_2 e^{x_2^2} \\ 2x_1 + \cos(x_1)x_2^2 & 2\sin(x_1)x_2 \\ \frac{2x_1}{1+x_1^2} & 0 \end{bmatrix} \end{aligned}$$

4 Linearization

Consider a mass attached to two springs:



We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$.

a) Write this model in state space form $\dot{x} = f(x)$.

Answer

We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right)\end{bmatrix}.$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Answer

We find the equilibrium by solving $0 = \dot{x} = f(x)$:

$$\begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right)\end{bmatrix}.$$

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

c) Linearize your model about the equilibrium.

Answer

$$\left.\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{x=(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - X_0\frac{a^2}{(x_1^2+a^2)^{3/2}}\right) & 0\end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0\end{bmatrix}$$

So the linearized system is

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} \vec{x}.$$

d) Compute the eigenvalues of your linearized model. Is this equilibrium stable?

Answer

To compute the eigenvalues, we solve

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{2k}{m}\left(1 - \frac{X_0}{a}\right).$$

Since $X_0 < a$, this means that $\left(1 - \frac{X_0}{a}\right) > 0$. So we have a pair of imaginary eigenvalues

$$\lambda = \pm\sqrt{\frac{2k}{m}\left(1 - \frac{X_0}{a}\right)}j.$$

Since the linearized system has purely imaginary eigenvalues that are not repeated, their real parts are zero. Therefore the equilibrium is unstable.

5 Stability in discrete time system

Determine which values of α and β will make the following discrete-time state space models stable. Assume, α and β are real numbers and $b \neq 0$.

a)
$$x(t+1) = \alpha x(t) + bu(t)$$

Answer

$$|\alpha| < 1$$

b)
$$\vec{x}(t+1) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}(t) + b\vec{u}(t)$$

Answer

The eigenvalues of this system are:

$$\lambda = \alpha \pm j\beta$$

$$|\lambda| = \sqrt{\alpha^2 + \beta^2}$$

For this system to be stable, $|\lambda| < 1$, so

$$\alpha^2 + \beta^2 < 1$$

c)
$$\vec{x}(t+1) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}(t) + b\vec{u}(t)$$

Answer

The eigenvalues of this system are

$$\lambda = 1, 1$$

This means that regardless of α, this system is always unstable since $|\lambda| \geq 1$.