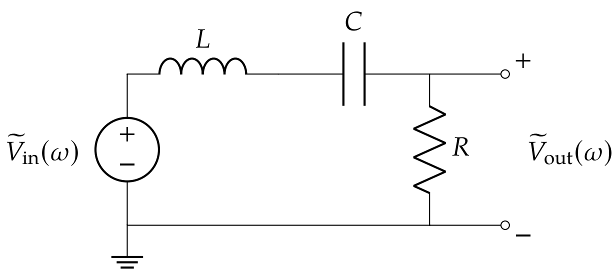
1 Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $X(\omega)$ is a real valued function of ω.**

Answer

Since the capacitor, resistor and inductor are in series, the equivalent impedance is given by,

$$Z_{RLC}(\omega) = R + Z_L(\omega) + Z_C(\omega)$$

$$\implies Z_{RLC}(\omega) = R + j\omega L + \frac{1}{j\omega C}$$

Since,

$$\frac{1}{j} = -j$$

$$Z_{RLC}(\omega) = R + j(\omega L - \frac{1}{\omega C})$$

Hence,

$$A(\omega) = R$$

and

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

b) **Write down the transfer function $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for this circuit.**

Answer

Using the same voltage divider rule we've used in the past, V_{out} is:

$$\widetilde{V_{out}} = \frac{\widetilde{V_{in}} R}{Z_{RLC}}$$

$$H(\omega) = \frac{\widetilde{V_{out}}}{\widetilde{V_{in}}} = \frac{R}{Z_{RLC}(\omega)}$$

$$H(\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})}$$

c) **At what frequency** ω_n **does** $X(\omega_n) = 0$**?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other.)

What happens to the relative magnitude of the impedances of the capacitor and inductor as ω **moves above and below** ω_n**? What is the value of the transfer function at this frequency?**

Answer

$$X(\omega_n) = 0 = \omega_n L - \frac{1}{\omega_n C}$$

Multiplying both sides by ω_n:

$$0 = \omega_n^2 L - \frac{1}{C}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

As the frequency increases above ω_n, the magnitude of the impedance of the inductor gets higher while the capacitor gets lower. Since the two components are in series, the impedance of the inductor will dominate. As the frequency decreases below ω_n, the magnitude of the impedance of the capacitor gets higher while the inductor gets lower. In this case the impedance of the capacitor will dominate.

At ω_n, $Z_{RLC} = R$, since the imaginary components cancel out perfectly. As a result

$$H(\omega_n) = 1$$

d) In most filters, we are interested in the cutoff frequency, since that helps define the frequency range over which the filter operates. Remember that this is the frequency at which the magnitude of the transfer function drops by a factor of $\sqrt{2}$ from its maximum value. Notice that the real part of the impedance Z_{RLC} is not changing with frequency and stays at R. What we care about is the frequencies where the imaginary part of the impedance equals $-jR$ to $+jR$.

To do this, we want to see what happens in the neighborhood of ω_n and so express the combined effect of the capacitor and inductor in terms of $\omega = \omega_n + \Delta\omega$, where $\Delta\omega$ is (presumably) a relatively small number compared to ω_n.

Write an expression for $X(\omega_n + \Delta\omega)$, where $\Delta\omega$ is a variable shift from ω_n. Find the values of $\Delta\omega_1$ and $\Delta\omega_2$ which give $X(\omega_n + \Delta\omega_1) = -R$, and $X(\omega_n + \Delta\omega_2) = +R$. Use the approximation that $\frac{1}{1+x} \approx 1 - x$ if $x \ll 1$.

Answer

From the first part we know that,

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

So we get,

$$X(\omega_n + \Delta\omega) = (\omega_n + \Delta\omega)L - \frac{1}{(\omega_n + \Delta\omega)C}$$

$$X(\omega_n + \Delta\omega) = \omega_n L + \Delta\omega L - \frac{1}{\omega_n C(1 + \frac{\Delta\omega}{\omega_n})}$$

Using, $\frac{1}{1+x} \approx 1 - x$ for $x << 1$ we get,

$$X(\omega_n + \Delta\omega) \approx \omega_n L + \Delta\omega L - \frac{1}{\omega_n C}(1 - \frac{\Delta\omega}{\omega_n})$$

At the resonance condition, $\omega_n L = \frac{1}{\omega_n C}$. As a result we get,

$$X(\omega_n + \Delta\omega) \approx \Delta\omega L + \frac{\Delta\omega}{\omega_n^2 C}$$

From part (c) $\frac{1}{\omega_n^2} = LC$ we get,

$$X(\omega_n + \Delta\omega) \approx 2\Delta\omega L$$

Hence, if $X(\omega_n + \Delta\omega_2) = R$, then we get,

$$\Delta\omega_2 = \frac{R}{2L}$$

Similarly, when $X(\omega_n + \Delta\omega_1) = -R$, we get,

$$\Delta\omega_1 = \frac{-R}{2L}$$

e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

Answer

At low frequencies,

$$X(\omega) \approx -\frac{1}{\omega C}$$

Plugging this back in to the original transfer function we get a CR high-pass filter.

$$H(\omega) \approx \frac{R}{R - j\frac{1}{\omega C}}$$

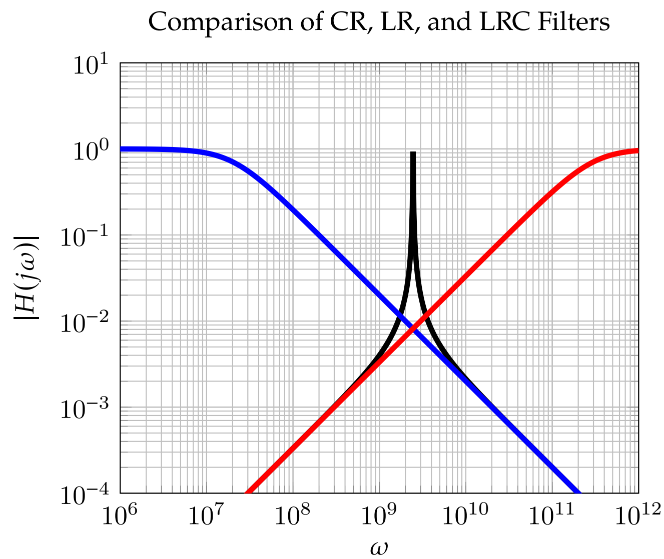
At high frequencies,

$$X(\omega) \approx \omega L$$

Plugging this back in to the original transfer function we get a LR low-pass filter.

$$H(\omega) \approx \frac{R}{R + j\omega L}$$

f) **A bode plot for a CR filter, a LR filter, and a LCR filter is shown for $L = 9nH$, $R = 0.18\Omega$, and $C = 18.7pF$. Assign each filter to its corresponding line on the plot.**
Label the locations of the corner frequencies, and describe the behavior of the LCR filter at very high and low frequencies.



Answer

The CR high pass filter is the red line, the LR blue, and the LCR black. Our band-pass filter looks like an LR low-pass filter at high frequencies and a CR high-pass filter at low frequencies. Note that in this case, the cutoff frequencies for the LR and CR filters are not the same as LCR cutoff frequencies or the resonance frequency. In the vicinity of the resonance frequency, notice that the slope of the filter is much larger than any first-order LR or RC filter. This allows for better rejection of frequencies outside of the desired frequency.

The corner frequency for the LR filter is $\frac{R}{L} = 20\text{Mrad/s}$ and for the CR filter is $\frac{1}{RC} = 300\text{Grad/sec}$. The two frequencies at which the magnitude of the LRC transfer function equals $\frac{1}{\sqrt{2}}$ are $2.4\text{Grad/s} \pm 10\text{Mrad/s}$.