1 Phasors

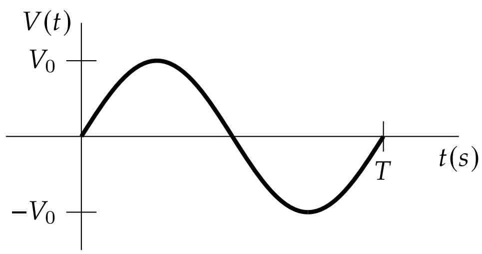
We consider sinusoidal voltages and currents of a specific form:

Voltage	$v(t) = V_0 \cos(\omega t + \phi_v)$
Current	$i(t) = I_0 \cos(\omega t + \phi_i)$

where,

1. V_0 is the voltage amplitude and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

2. ω is the angular frequency of oscillation. ω is related to frequency by $\omega = 2\pi f$. Frequency f is the number of oscillation cycles that occur per second. If T is the *period* of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is $f = \frac{1}{T}$.



3. ϕ_v and ϕ_i are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \text{Re}\left(e^{j\theta}\right) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = \text{Re}\left(V_0 e^{j(\omega t + \phi_v)}\right) = \text{Re}\left(V_0 e^{j(\omega t + \phi_v)}\right) = \text{Re}\left(V_0 e^{j\phi} e^{j\omega t}\right) \tag{1}$$

The part of this equation that does not depend on time is called the phasor[1]:

$$\widetilde{V} = V_0 e^{j\phi_v}$$

[1] If you look at notes from previous offerings of the course, you might see $\widetilde{V} = \frac{1}{2}V_0 e^{j\phi_V}$ as the definition of a phasor. This definition is different from what we will use in this course, but leads to the same impedance and time domain solutions

The complex conjugate of the phasor is given by:

$$\overline{\widetilde{V}} = V_0 e^{-j\phi_v}$$

The phasor representation is a constant that contains the magnitude and phase information of the signal. The time-varying part of the signal does not need to be explicitly represented, because it is given by $e^{j\omega t}$, which is always implicit when using phasors. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like (16A style) circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

Note: We can only use phasors if we know that all of our signals are sinusoids. We will see later in 16B (Module 4) why this is not too restrictive a condition.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\widetilde{V}$ is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \text{Re}(\widetilde{V} e^{j\omega t}) = \frac{1}{2}\left(\widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t}\right)$$

The standard forms for voltage and current phasors are given below:

Voltage	$\widetilde{V} = V_0 e^{j\phi_v}$
Current	$\widetilde{I} = I_0 e^{j\phi_i}$

We define the *impedance* of a circuit component to be $Z = \frac{\widetilde{V}}{\widetilde{I}}$, where $\widetilde{V}$ and $\widetilde{I}$ represent the voltage across and the current through the component, respectively. For capacitors and inductors, this impedance will generally depend on $j\omega$.

Phasor Relationship for Resistors

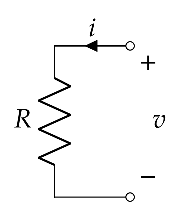


Figure 1: A simple resistor circuit

Consider a simple resistor circuit as in Figure 1, with current being

$$i(t) = I_0 \cos(\omega t + \phi) = \frac{I_0}{2} e^{j\phi} e^{j\omega t} + \frac{I_0}{2} e^{-j\phi} e^{-j\omega t} = \text{Re}\left(I_0 e^{j\phi} e^{j\omega}\right)$$

We see that $\widetilde{I} = I_0 e^{j\phi}$. By Ohm's law,

$$\begin{aligned} v(t) &= i(t)R \\ &= I_0 R\cos\left(\omega t + \phi\right) = \frac{I_0 R}{2} e^{j\phi} e^{j\omega t} + \frac{I_0 R}{2} e^{-j\phi} e^{-j\omega t} \\ &= \text{Re}\left(I_0 R e^{j\phi} e^{j\omega}\right) \end{aligned}$$

Hence $\widetilde{V} = I_0 R e^{j\phi}$ In the phasor domain,

$$\widetilde{V} = R\widetilde{I}$$

We usually refer to the impedance of the resistor, Z_R, in the phasor domain. Since $Z_R = R$, we can also write:

$$\widetilde{V} = Z_R \widetilde{I}$$

Phasor Relationship for Capacitors

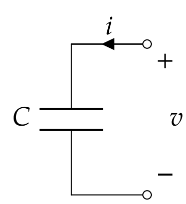


Figure 2: A simple capacitor circuit

Consider a capacitor circuit as in Figure 2, with voltage being

$$v(t) = V_0 \cos\left(\omega t + \phi\right)$$

By the capacitor equation, and expanding cos in the complex exponential domain:

$$\begin{aligned} v(t) &= V_0 \cos(\omega t + \phi) \\ &= \frac{V_0}{2} e^{j(\omega t + \phi)} + \frac{V_0}{2} e^{-j(\omega t + \phi)} \\ &= \frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \\ &= \frac{1}{2} \left(\widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t} \right) \end{aligned}$$

$$\begin{aligned} i(t) &= C \frac{d}{dt} v(t) \\ &= C \frac{d}{dt} \left(\frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \right) \\ &= j\omega C \frac{V_0}{2} e^{j\phi} e^{j\omega t} - j\omega C \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \\ &= \frac{1}{2} \left(\widetilde{I} e^{j\omega t} + \overline{\widetilde{I}} e^{-j\omega t} \right) \end{aligned}$$

where

$$\widetilde{I} = j\omega C V_0 e^{j\phi} = j\omega C \widetilde{V}$$

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted Z_C, which is given by:

$$Z_C = \frac{\widetilde{V}}{\widetilde{I}} = \frac{1}{j\omega C}.$$

2 Inductor Impedance

Given the voltage-current relationship of an inductor $V = L\frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

Answer

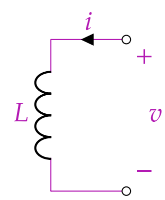


Figure 3: A simple inductor circuit

Consider a simple resistor circuit as in Figure 3, with current being

$$i(t) = I_0 \cos(\omega t + \phi)$$

By the inductor equation,

$$\begin{aligned} v(t) &= L\frac{\mathrm{d}i}{\mathrm{d}t}(t) \\ &= -LI_0\omega \sin(\omega t + \phi) \\ &= LI_0\omega \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \\ &= (\omega L)I_0 \cos\left(\omega t + \phi + \frac{\pi}{2}\right) \end{aligned}$$

In the phasor domain,

$$\widetilde{V} = \omega L e^{j\frac{\pi}{2}}\widetilde{I} = j\omega L\widetilde{I}$$

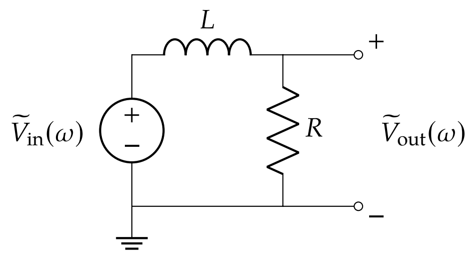
The impedance of an inductor is an abstraction to model the inductor as a resistor in the phasor domain. This is denoted Z_L.

$$Z_L = \frac{\widetilde{V}}{\widetilde{I}} = j\omega L$$

3 LC transfer function

Determine $H(j\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for the following circuit, and use the transfer function you find to describe how the circuit responds to low and high frequencies.

a) Determine $H(j\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$. How does this circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?



[H]

Answer

The strategy is to use the voltage divider formula, *i.e.*,

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}.$$

$$H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}}. \tag{2}$$

At low frequencies, we have

$$\lim_{\omega \to 0} H(j\omega) = 1, \tag{3}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} H(j\omega) = 0. \tag{4}$$

So this circuit is a *low-pass filter*.

We often plot transfer functions as *bode plots* for magnitude and phase. For $L = 10\,\mu\text{H}$ and $R = 10\,\text{k}\Omega$ the frequency response is as follows:

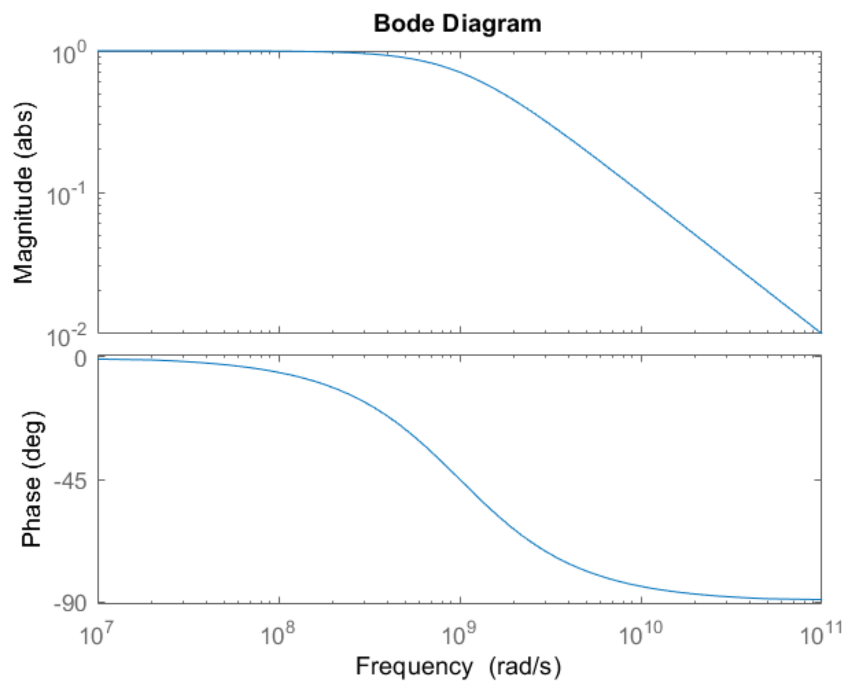


Figure 4: Magnitude and phase for *LR* low pass filter. Note that the magnitude is ploted in a log-log scale and the phase is ploted in a semi-log scale.