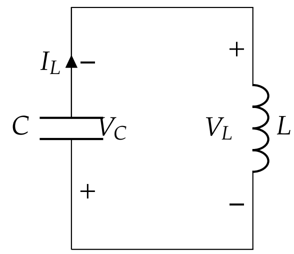
1 LC Tank: Diagonalization with complex eigenvalues

Consider the following circuit like you saw in lecture:



This is sometimes called an LC tank and we will derive its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $\frac{dV_C}{dt}(t = 0) = 0$.

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, and C.**

Answer

For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{1}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{2}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_2(t) = \frac{1}{C}x_1(t), \tag{3}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_L(t) = 0. \tag{4}$$

Using the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + L\frac{d}{dt}I_L(t) = 0, \tag{5}$$

which we can rewrite as

$$\frac{d}{dt} I_L(t) = -\frac{1}{L} V_C(t). \tag{6}$$

If we use the state variable names, this becomes

$$\frac{d}{dt} x_1(t) = -\frac{1}{L} x_2(t), \tag{7}$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt} x_1(t) = -\frac{1}{L} x_2(t) \tag{8}$$

$$\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t). \tag{9}$$

b) **Write the system of equations in vector/matrix form with the vector state variable** $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$**.** This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Find the initial conditions $\vec{x}(0)$.

Answer

By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{10}$$

which is in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with

$$A = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \tag{11}$$

We know that $V_C(0) = V_S$ and we know that $i_L(0) = C\frac{d}{dt}V_C(0) = 0$, thus $\vec{x}(0) = \begin{bmatrix} 0 \\ V_S \end{bmatrix}$

c) **Find the eigenvalues of the** A **matrix symbolically.**

Answer

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix}\right) \tag{12}$$

$$= \lambda^2 + \frac{1}{LC} = 0. \tag{13}$$

Solving for λ we see

$$\lambda = \pm \frac{j}{\sqrt{LC}}. \tag{14}$$

d) **What are the eigenvectors associated with these eigenvalues?**

Answer

For $\lambda_1 = -\frac{j}{\sqrt{LC}}$ the eigenvector $\vec{v}_1$ must span the nullspace of $A - \lambda_1 I$. That is,

$$(A - \lambda_1 I)\vec{v}_1 = \begin{bmatrix} \frac{j}{\sqrt{LC}} & -\frac{1}{L} \\ \frac{1}{C} & \frac{j}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving we see that $b = aj\sqrt{\frac{L}{C}}$ so we pick

$$\vec{v}_1 = \begin{bmatrix} j \\ -\sqrt{\frac{L}{C}} \end{bmatrix}$$

We can solve for $\vec{v}_2$ associated with $\lambda_2 = \frac{j}{\sqrt{LC}}$ the same way, or we can note that for a real matrix ($\overline{A} = A$), if $A\vec{v} = \lambda\vec{v}$, taking the complex conjugate of both sides we have

$$\overline{A\vec{v}} = \overline{\lambda\vec{v}}$$
$$\overline{A}\overline{\vec{v}} = \overline{\lambda}\overline{\vec{v}}$$
$$A\overline{\vec{v}} = \overline{\lambda}\overline{\vec{v}}$$

Thus, since $\lambda_2 = \overline{\lambda_1}$ we must have

$$\vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} -j \\ -\sqrt{\frac{L}{C}} \end{bmatrix}$$

Note that we can take any non-zero scalar multiples of $\vec{v}_1$ and $\vec{v}_2$ as the eigenvectors.

e) **Use the eigenvalues and eigenvectors found above to diagonalize A as $A = V\Lambda V^{-1}$ where Λ is a diagonal matrix. Suppose $L = 9\,\mathrm{nH}$ and $C = 1\,\mathrm{nF}$.**

Answer

We solve symbolically and then plug in the desired values: Let

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} j & -j \\ -\sqrt{\frac{L}{C}} & -\sqrt{\frac{L}{C}} \end{bmatrix}.$$

Then

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} -\sqrt{\frac{L}{C}} & j \\ \sqrt{\frac{L}{C}} & j \end{bmatrix} = \frac{1}{-2j\sqrt{\frac{L}{C}}} \begin{bmatrix} -\sqrt{\frac{L}{C}} & j \\ \sqrt{\frac{L}{C}} & j \end{bmatrix} = \begin{bmatrix} -\frac{j}{2} & -\frac{1}{2}\sqrt{\frac{C}{L}} \\ \frac{j}{2} & -\frac{1}{2}\sqrt{\frac{C}{L}} \end{bmatrix}$$

For $A = V\Lambda V^{-1}$ we then have $\Lambda = V^{-1}AV = \begin{bmatrix} \frac{-j}{\sqrt{LC}} & 0 \\ 0 & \frac{j}{\sqrt{LC}} \end{bmatrix}$. Note that the diagonal values of Λ are the eigenvalues of A, as expected.

Replacing given values for L and C

$$V = \begin{bmatrix} j & -j \\ -3\,\Omega & -3\,\Omega \end{bmatrix} \quad V^{-1} = \begin{bmatrix} -\frac{j}{2} & -\frac{1}{6}\,\Omega^{-1} \\ \frac{j}{2} & -\frac{1}{6}\,\Omega^{-1} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -\frac{j}{3}\,\mathrm{GHz} & 0 \\ 0 & \frac{j}{3}\,\mathrm{GHz} \end{bmatrix}$$

f) Use a change of basis for the state variable $\vec{x}(t)$ into $\vec{z}(t)$ such that $\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t)$, and express the initial conditions $\vec{z}(0)$
Solve the differential equations in $\vec{z}(t)$

Answer

Let $\vec{z}(t) = V^{-1}\vec{x}(t)$. Since $\vec{x}(t) = V\Lambda V^{-1}\vec{x}(t)$, we have $V^{-1}\vec{x}(t) = \Lambda V^{-1}\vec{x}(t)$, or equivalently $\frac{d}{dt}\vec{z}(t) = \Lambda\vec{z}(t)$, as desired.
Thus

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-j}{\sqrt{LC}} & 0 \\ 0 & \frac{j}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}.$$

For the initial conditions, we have

$$\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{j}{2} & -\frac{1}{2}\sqrt{\frac{C}{L}} \\ \frac{j}{2} & -\frac{1}{2}\sqrt{\frac{C}{L}} \end{bmatrix} \begin{bmatrix} 0 \\ V_S \end{bmatrix} = \frac{V_s}{2}\sqrt{\frac{C}{L}} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

From this we have solutions $z_1(t) = -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{-\frac{j}{\sqrt{LC}}t}$ and $z_2(t) = -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{\frac{j}{\sqrt{LC}}t}$.

Note that $z_1(t) = \overline{z_2(t)}$ In vector form:

$$\vec{z}(t) = \begin{bmatrix} -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{-\frac{j}{\sqrt{LC}}t} \\ -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{\frac{j}{\sqrt{LC}}t} \end{bmatrix}.$$

Using the desired values:

$$\vec{z}(t) = \begin{bmatrix} -(166.7\,\mathrm{mA})e^{-\frac{j}{3\,\mathrm{ns}}t} \\ -(166.7\,\mathrm{mA})e^{\frac{j}{3\,\mathrm{ns}}t} \end{bmatrix}.$$

g) **Convert your solutions back to $\vec{x}(t)$. Plot $V_C(t)$ and $I_L(t)$.** What do you notice about the solutions? Are they complex functions? HINT: Remember $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Answer

We have

$$\begin{aligned}
\vec{x}(t) = V\vec{z}(t) &= \begin{bmatrix} j & -j \\ -\sqrt{\frac{L}{C}} & -\sqrt{\frac{L}{C}} \end{bmatrix} \begin{bmatrix} -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{-\frac{j}{\sqrt{LC}}t} \\ -\frac{V_s}{2}\sqrt{\frac{C}{L}}e^{\frac{j}{\sqrt{LC}}t} \end{bmatrix}. \\
&= \begin{bmatrix} -j\frac{V_s}{2}\sqrt{\frac{C}{L}}\left(e^{-\frac{j}{\sqrt{LC}}t} - e^{\frac{j}{\sqrt{LC}}t}\right) \\ \frac{V_S}{2}\left(e^{-\frac{j}{\sqrt{LC}}t} + e^{\frac{j}{\sqrt{LC}}t}\right) \end{bmatrix} \\
&= \begin{bmatrix} -V_s\sqrt{\frac{C}{L}}\sin\left(\frac{t}{\sqrt{LC}}\right) \\ V_s\cos\left(\frac{t}{\sqrt{LC}}\right) \end{bmatrix} \qquad \text{Since} \qquad \begin{aligned} \sin(j\theta) &= \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}) \\ \cos(j\theta) &= \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \end{aligned}
\end{aligned}$$

Using the provided values:

$$I_L(t) = -I_{\max}\sin\left(\frac{t}{3\,\mathrm{ns}}\right), \qquad I_{\max} = 333.33\,\mathrm{mA}$$

$$V_C = V_S\cos\left(\frac{t}{3\,\mathrm{ns}}\right), \qquad V_S = 1\,\mathrm{V}$$

2 Complex Matrix Inverse

Consider a complex matrix

$$M = M_r + jM_i$$

and its inverse

$$N = N_r + jN_i$$

a) **Show that the inverse of $\overline{M} = M_r - jM_i$ (the complex conjugate of M) is equal to $\overline{N} = N_r - jN_i$ (the complex conjugate of N).**

Answer

Because N is the inverse of M, we know

$$\begin{aligned} I = NM &= (N_r + jN_i)(M_r + jM_i) \\ &= (N_r M_r - N_i M_i) + j(N_r M_i + N_i M_r) \\ &\Rightarrow I = N_r M_r - N_i M_i \qquad 0 = N_r M_i + N_i M_r \end{aligned}$$

Note that we isolated the real and imaginary terms in the product. Because the product of a matrix and its inverse must be the identity matrix, the real terms must equal the identity matrix, and the imaginary terms must be 0.

Similarly, we try evaluating the product of $\overline{NM}$:

$$\begin{aligned} \overline{NM} &= (N_r - jN_i)(M_r - jM_i) \\ &= (N_r M_r - N_i M_i) - j(N_r M_i + N_i M_r) \\ &= I - j0 \\ &= I \end{aligned}$$

Thus, $\overline{M}^{-1} = \overline{N} = \overline{M^{-1}}$.

The inverse of the complex conjugate of a matrix is equal to the complex conjugate of the matrix's inverse.