Continuous and discrete time

There are two different dialects for modeling change over time. Thus far we have modeled real-life events using differential equations and initial conditions. For example, the voltage across a capacitor connected to a voltage source by a resistor is fully described by the following differential equation and initial conditions.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} v_{\text{in}}(t), \quad v_C(0) = v_0 \tag{1}$$

Abstracting away particulars, *continuous-time* scalar linear systems can be represented in variants of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + \mu u(t), \quad x(0) = x_0. \tag{2}$$

This discussion will introduce *discrete-time* scalar linear systems, which have models similar to the following:

$$x[t+1] = ax[t] + bu[t], \quad x[0] = x_0. \tag{3}$$

Notice that evolution is represented by defining the transition from $x[t]$ to $x[t+1]$. The state x is not a continuous function of time, but a sequence of individual moments. Can you think of systems in life that are naturally more susceptible to discrete-time modeling?

1 Differential equations with piecewise constant inputs

1. Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda \left(x(t) - u(t) \right). \tag{4}$$

Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{5}$$

Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \leq t < (n+1)T. \tag{6}$$

For a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

Answer

As $u = u[n]$ is constant between samples $x[n]$ and $x[n+1]$, the following differential equation and initial conditions describe what is happening to x during this interval:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda(x(t) - u[n]), \quad x(nT) = x[n]. \tag{7}$$

Let's use the entire RHS of the differential equation as a change of variables from x to z. (Other changes of variables are possible, e.g. $z_{\text{alt}} = x(t) - u(t)$.)

$$z(t) = \lambda(x(t) - u[n]) \tag{8}$$

By differentiating both sides of this relationship, we can achieve a differential equation for z.

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\lambda(x(t) - u[n])\right) \tag{9}$$

$$= \lambda \frac{\mathrm{d}}{\mathrm{d}t} x(t) - \lambda \frac{\mathrm{d}}{\mathrm{d}t} u[n] \tag{10}$$

As $u[n]$ is a constant, $\frac{\mathrm{d}}{\mathrm{d}t} u[n] = 0$.

$$= \lambda \frac{\mathrm{d}}{\mathrm{d}t} x(t) \tag{11}$$

Apply the differential equation for x.

$$= \lambda \left(\lambda(x(t) - u[n])\right) \tag{12}$$

We can recognize the expression in the parentheses as $z(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \lambda z(t) \tag{13}$$

A solution to this equation is of the form $z(t) = Ke^{\lambda t}$, where K is a constant to be determined. By reversing Eqn. 8, we arrive at a solution for $x(t)$—where, still, K remains to be determined.

$$x(t) = \frac{1}{\lambda} z(t) + u[n] \tag{14}$$

$$= \frac{1}{\lambda} Ke^{\lambda t} + u[n] \tag{15}$$

We will determine K by insisting that our solution comply with the initial conditions of Eqn. 7.

$$x(nT) = \frac{1}{\lambda} Ke^{\lambda(nT)} + u[n] = x[n] \tag{16}$$

$$K = \frac{\lambda}{e^{\lambda(nT)}} \left(x[n] - u[n]\right) \tag{17}$$

Now we have enough to evaluate $x((n+1)T)$, by evaluating $x(t)$ at $(n+1)T$.

$$x((n+1)T) = \frac{1}{\lambda} \left(Ke^{\lambda(n+1)T}\right) + u[n] \tag{18}$$

$$= \frac{1}{\lambda} \left(\frac{\lambda}{e^{\lambda(nT)}} \left(x[n] - u[n]\right) e^{\lambda(n+1)T}\right) + u[n] \tag{19}$$

$$= e^{\lambda T} x[n] + \left(1 - e^{\lambda T}\right) u[n] \tag{20}$$

Rewrite $x((n+1)T)$ as $x[n+1]$:

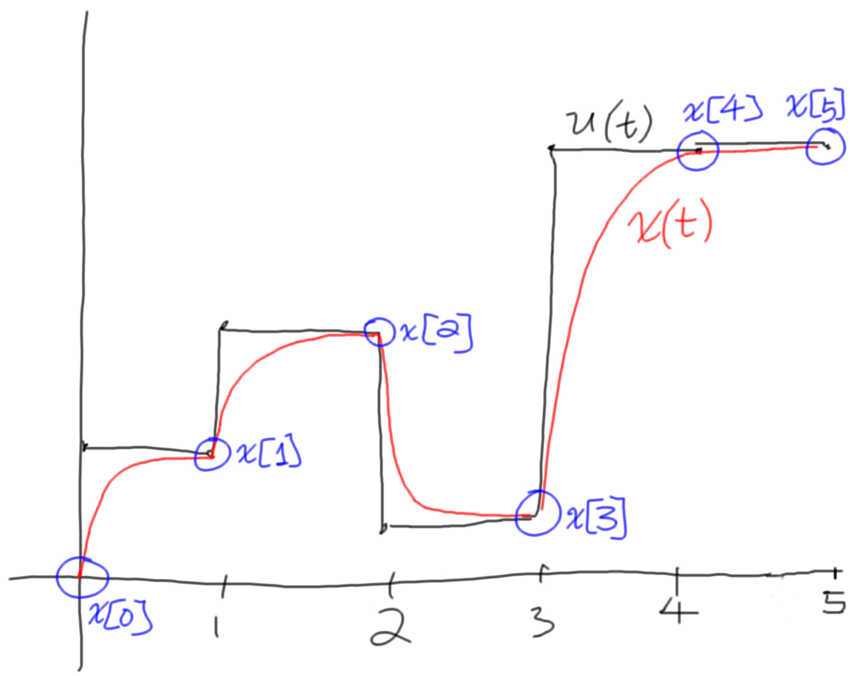
$$x[n+1] = e^{\lambda T} x[n] + \left(1 - e^{\lambda T}\right) u[n] \tag{21}$$

and we are done.

2. Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then sketch $x(t)$. Mark $x[n]$. Your sketch doesn't have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that λT is large and negative?

Answer

A typical drawing might look similar to this:



Notice that the displacement between $x(t)$ and its moving target $u(t)$ is always in exponential decay (it is proportional to $z(t)$). Because λT is large and negative, $e^{\lambda T} \approx 0$, so

$$x[n+1] = e^{\lambda T} x[n] + \left(1 - e^{\lambda T}\right) u[n] \tag{22}$$

$$\approx u[n] \tag{23}$$

3. Let $T = 1$ and $\lambda = -1$. Define $u[n]$ as follows:

$$u[n] = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}. \tag{24}$$

Sketch $x(t)$.

Answer

Notice how u, which is x's target, is flipping so quickly that x never gets close to the finish line. It gets partway there and then is told to turn around. An approximate sketch (with features exaggerated) would look like this: